JOINT INSTITUTE FOR NUCLEAR RESEARCH
The Bogoliubov Laboratory of Theoretical Physics (BLTP)

# FINAL REPORT ON THE START PROGRAMME

## Asymptotic Symmetries in Gravitational Theory and Bondi-Metzner-Sachs Group

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# 1 Abstract

The modern theory of gravity is described by the general relativity, which is mathematically based on the idea of our physical space as spacetime - a four-dimensional manifold. The basic principle of the theory is described by Einstein's field equations. They can be popularly explained as follows - mass tells space how to bend, and curved space tells matter how to move. From the special relativity we know that equations are invariant under Lorentz transformations.

But the definition of the inhomogeneous Lorentz group (Poincaré group) as a symmetry group breaks down in the presence of gravitational fields, even if the dynamic effects of gravity are negligible.

In 1962, R. Sachs tried to derive the Lorentz group as an "asymptotic symmetry group" that leaves unchanged the form of boundary conditions suitable for asymptotically flat gravitational fields. After analyzing the work of Bondi and Metzner on gravitational radiation, R. Sachs showed that for reasonable boundary conditions, not the Lorentz group is obtained, but a broader group, which is now commonly called the Bondi-Metzner-Sachs group. The practice is devoted to acquaintance with the BMS group and asymptotic symmetries.

# 2 Lorentz transformations

To define Lorentz transformations, let's recall some facts from differential geometry. Further, the summation is performed according to Einstein's rule and the speed of light is assumed to be normalized $c = 1$. The interval (distance between two infinitely close points in the spacetime) can be written using a metric tensor $g_{\alpha\beta}$:

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta, \quad \alpha,\beta = 0,...,3,$$

where the coordinates are indicated $x^0 = t$, $x^1 = x$, $x^2 = y$, $x^3 = z$. In a locally inertial reference frame, the metric tensor $g_{\alpha\beta}$ is equal to the Minkowski metric $\eta_{\alpha\beta}$:

$$g_{\alpha\beta} = \eta_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Lorentz transformations are a transition from one coordinate system $x^\alpha$ to another system $x'^\alpha$, defined according to the rule:

$$x'^\alpha = \Lambda^\alpha_\beta x^\beta + a^\alpha,$$

where $\Lambda$ - an element of the Lorentz group defined by the condition:

$$ds'^2 = \eta_{\alpha'\beta'} dx^{\alpha'} dx^{\beta'} = \eta_{\mu\nu} \Lambda^\mu_{\alpha'} \Lambda^\nu_{\beta'} dx^{\alpha'} dx^{\beta'},$$

or briefly: $\Lambda^\mu_{\alpha'} \Lambda^\nu_{\beta'} \eta_{\mu\nu} = \eta_{\alpha'\beta'}$.

The vector $a$ sets the shift (translation) and if it is nonzero, then such inhomogeneous Lorentz group is called a Poincare group. The Lorentz group is a subgroup of the Poincare group, it forms the following symmetries: inversion of space and time, their combination, rotations and boosts. Boost is a transition to a moving inertial frame of reference, without purely spatial rotation, shift and reflections.

In a component-by-component form, the transformations can be written as a system:

$$\begin{cases} t = x'sh(\phi) + t'ch(\phi) = \frac{x'+vt'}{\sqrt{1-v^2}}, \\ x = x'ch(\phi) + t'sh(\phi) = \frac{vx'+t'}{\sqrt{1-v^2}}, \\ y = y', \\ z = z'. \end{cases}$$

Hence, the Lorentz transformation matrix is given as the Jacobi matrix of the corresponding coordinate system:

$$\Lambda^a_{\alpha'} = \frac{\partial x^\alpha}{\partial x^{\alpha'}} = \begin{pmatrix} \frac{1}{\sqrt{1-v^2}} & \frac{v}{\sqrt{1-v^2}} & 0 & 0 \\ \frac{v}{\sqrt{1-v^2}} & \frac{1}{\sqrt{1-v^2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

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# 3 Linearized gravity

The main equation of general relativity is Einstein's field equation:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu}. \tag{1}$$

Here we want to find out the equations of motion obeyed by the perturbations of metric. In the neighborhood of the point $g_{\mu\nu}$ can be represented as the sum of $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $h_{\mu\nu}$ are linear perturbations of the Minkowski metric.

To obtain linearized equation, we rewrite the Christoffel symbols $\Gamma$, the Ricci tensor $R_{\mu\nu}$ and the Ricci scalar $R$:

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2}g^{\sigma\lambda}(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu}) = \frac{1}{2}\eta^{\sigma\lambda}(\partial_\mu h_{\nu\lambda} + \partial_\nu h_{\lambda\mu} - \partial_\lambda h_{\mu\nu}) + O(h^2).$$

Ricci tensor we obtain from Riemann tensor in linear order:

$$R_{\mu\nu\rho\sigma} = \eta_{\mu\lambda}\partial_\rho\Gamma^\lambda_{\nu\sigma} - \eta_{\mu\lambda}\partial_\sigma\Gamma^\lambda_{\nu\rho} = \frac{1}{2}(\partial_\rho\partial_\nu h_{\mu\sigma} - \partial_\rho\partial_\mu h_{\nu\sigma} - \partial_\sigma\partial_\nu h_{\mu\rho} + \partial_\sigma\partial_\mu h_{\nu\rho}),$$

$$R_{\mu\nu} = \eta^{\rho\sigma}R_{\mu\nu\rho\sigma} = \frac{1}{2}(\partial_\sigma\partial_\mu h^\sigma_\nu + \partial_\sigma\partial_\nu h^\sigma_\mu - \partial_\mu\partial_\nu h - \Box h_{\mu\nu}),$$

$$R = \eta^{\mu\nu}R_{\mu\nu} = \partial_\mu\partial_\nu h^{\mu\nu} - \Box h.$$

Let's substitute this into the right side of the equation (1) and use harmonic gauge (Lorentz gauge) $\partial_\alpha h^\alpha_\beta = \frac{1}{2}\partial_\beta h$, where $h$ is a trace of perturbation matrices, defined by formulas $\eta^{\mu\nu}h_{\mu\nu} = h$ or $h^\alpha_\alpha = h$:

$$R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R = \frac{1}{2}(-\Box h_{\mu\nu} + \eta_{\mu\nu}\Box h) = 8\pi G T_{\mu\nu}.$$

Simplifying the equation above and assuming the absence of a matter field ($T_{\mu\nu} = 0$), we get the wave equation:

$$\Box h_{\mu\nu} = 0, \tag{2}$$

where $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$ is d'Alembert operator.

Thus, weak perturbations of the metric against the background of flat spacetime can exist in the form of propagating self-sustaining waves.

Since the equation (2) is linear, the general solution can be constructed as a linear superposition of plane waves, so $h_{\mu\nu}(x) = \epsilon_{\mu\nu}(k)e^{ikx}$, where $\epsilon_{\mu\nu}$ is the polarization tensor and $k$ is a wave vector. Because of the harmonic gauge and symmetries, gravitational waves come in only 2 polarizations, just like electromagnetic waves. So the polarization tensor is characterised by 2 real numbers $\epsilon_+$, $\epsilon_\times$:

$$\epsilon_{\mu\nu} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \epsilon_+ & \epsilon_\times & 0 \\ 0 & \epsilon_\times & -\epsilon_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The two polarizations, found here for a classical gravitational wave, correspond to the two helicity states of the graviton in quantum field theory. And this allowed to make an assumption about existence of gravitational waves and gravitational radiation.

# 4 Bondi-Metzner-Sachs Group

So, if gravitational fields are negligible, two approaches for definition of Lorentz transformations can be distinguished[2]:

1)Lorentz transformations should leave the basic differential equations of physics invariant;

2)Lorentz transformations are symmetry transformations that preserve the numerical value of the metric tensor. If the effects of gravitational waves are taken into account according to general relativity theory, then any coordinate transformation will satisfy the first understanding, but in general it will not satisfy the second. Thus, only a homogeneous Lorentz group can be defined correctly, since these transformations are related to local properties of space.

But in theoretical physics, the inhomogeneous Lorentz group justifies conservation laws and defines possible types of elementary particles. Hence, the Lorentz transformations must be "asymptotic symmetries" and a new representation in an asymptotically flat spacetime with a modified Minkowski metric should be considered. Also, these transformations must satisfy some boundary conditions imposed on gravitational waves at infinity.

In Bondi's works, a solution to this problem was proposed, which was subsequently generalized in the Petrov-Pirani classification. The essence of the approach is to assume the absence of a priori knowledge about the properties of the defined group of asymptotic symmetry and to construct such a group using reasonable boundary conditions. The key role from the mathematical side is played by null (light-like) hypersurfaces.

The characteristic hypersurfaces of a given hyperbolic equation can be given in terms of the retarded time $u$ and the leading time $v$:

$$u = t - r, \quad v = t + r, \quad r^2 = \delta_{\alpha\beta}x^\alpha x^\beta,$$

that is, these are hypersurfaces along which the wave front moves. They are also null hypersurfaces, meaning their normals are null:

$$k_\alpha = -\partial_\alpha u, \quad n_\alpha = -\partial_\alpha v$$

$$\eta^{\alpha\beta}k_\alpha k_\beta = \eta_{\alpha\beta}n_\alpha n_\beta = 0$$

It follows from this that the normal direction is also tangent to the hypersurfaces, that is, the vector $k^\alpha = \eta^{\alpha\beta} k_\beta$ is tangent to $u = const$ hypersurface. Curves tangent to $k_\alpha$, these are null geodesic rays generating outgoing null hypersurfaces $u = const$. Using the family of outgoing rays, gravitational waves can be described. Such a construction will be one of the necessary conditions for the existence of the Bondi-Metzner-Sachs metric.

Let the space be defined as the topological product of a real axis and a two-dimensional unit sphere and this space is smoothly covered by three coordinates $\theta$,$\phi$,$u$, where $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$, $-\infty \leq u \leq \infty$. Let's define the Bondi-Metzner-Sachs transformation:

$$\theta' = \theta'(\theta,\phi,u), \quad \phi' = \phi'(\theta,\phi,u), \quad u' = K(\theta,\phi)(u - \varsigma(\theta,\phi)),$$

where $(\theta,\phi) \rightarrow (\theta',\phi')$ - conformal transformation of a sphere into itself, $K$ is a conformal factor:

$$d\theta'^2 + sin^2\theta'^2 d\phi'^2 = K^2(\theta^2 + sin^2\theta d\phi^2)$$

The fig. (1) shows the geometric properties of the coordinates in the generic case.

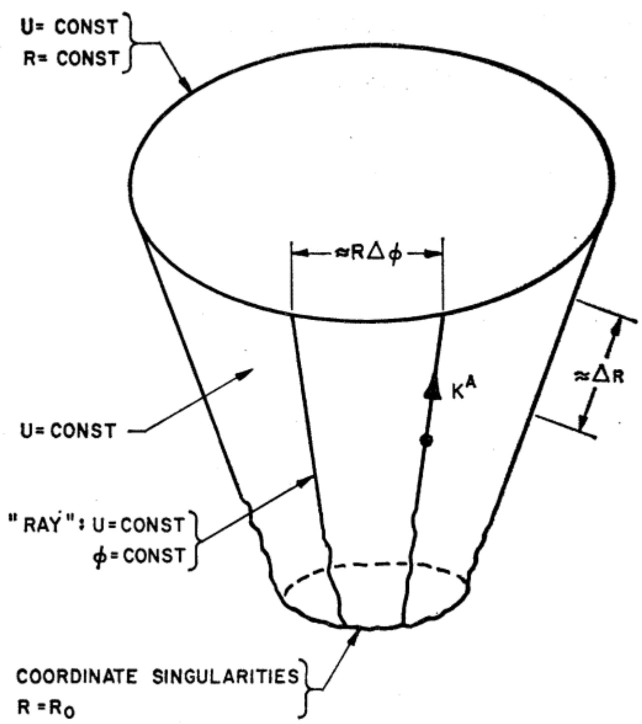


Figure 1: Bondi-Metzner-Sachs coordinate system defined at a timelike worldtube with a null cone. $R = R_0$ is a cut of worldtube, where $u$ varies along surface geodesics. Hypersurfaces $u, \phi = const$ correspond to null rays, $u = const$ is a null cone

Transformations $\theta' = \theta$, $\phi' = \phi$, $u' = u + \varsigma$ are called supertranslations, it is a subgroup of Bondi-Metzner-Sachs group. The $\varsigma$ function can be decomposed into spherical harmonics:

$$\varsigma = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{m,l} Y_{l,m}(\theta,\phi),$$

where $a_{m,l}$ is an infinite set of constants defining supertranslations. If $a_{m,l} = 0$ for $l > 2$, then $\varsigma$ is represented by:

$$\varsigma = \epsilon_0 + \epsilon_1 sin\theta cos\phi + \epsilon_2 sin\theta sin\phi + \epsilon_3 cos\theta.$$

In this case, the transformation is called translation with 4 parameters and is a subgroup of supertranslations. A fairly detailed introduction to can be found in the book [3]

Let the Bondi-Sachs coordinates be constructed on a family of null hypersurfaces $u = const$ and represented as $x^\alpha = (u, r, x^A)$, where $x^A$ are the angular coordinates $\theta$, $\phi$. Using the definition (2) and under suitable boundary conditions, which will be discussed below, it is possible to write the interval in general form:

$$ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta = -\frac{V}{r}e^{2\psi}du^2 - 2e^{2\psi}dudr + r^2h_{AB}(dx^A - U^A du)(dx^B - U^B du), \quad (3)$$

where the indices $A$,$B$ run through two values, the functions $V(u, \theta, \phi)$, $\psi(u, \theta, \phi)$, $U(\ \theta, \phi)$, $det(h_{AB}) = b(u, \theta, \phi)$ are arbitrary functions of their arguments and $g_{AB} = r^2h_{AB}$. At the same time, restrictions must be satisfied:

$$u_0 \leq u \leq u_1, \quad r_0 \leq r \leq \infty, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \phi \leq 2\pi. \quad (4)$$

Necessary conditions for the existence of a metric (3):

1)Hypersurfaces $u = const$ are tangent to the light cone everywhere. The normal vector $k_\alpha = -\partial_\alpha u$ satisfies the equality $g^{\alpha\beta}\partial_\alpha u\partial_\beta u = 0$, hence $g^{uu} = 0$ and the vector $k^\alpha = -g^{\alpha\beta}\partial_b u$ remains tangent to the null rays.

2) Variable $r$ is the radius of the light cone section. In the Newman-Penrose formalism, the affine parameter $\lambda$ is used instead of the radial coordinate, since the expansion of the $\Theta$ null hypersurface goes to infinity at $r = 0$:

$$\Theta = \nabla_\alpha(e^{-2\psi}k^\alpha) = \frac{2}{r}e^{-2\psi}, \quad k^\alpha\partial_\alpha = -g^{ur}\partial_r$$

The relation of the radius and the affine parameter is expressed by the formula: $\partial_\alpha\lambda = e^{2\psi}$.

3)The variables $\theta$, $\phi$ are constant along each ray forming a cone, that is, $k^\alpha\partial_\alpha x^A = -g^{\alpha\beta}(\partial_\alpha u)(\partial_\beta x^A)$. The ray is defined as a line with a tangent vector $k^\alpha = -(\partial_\beta u)g^{\alpha\beta}$.

The study of field equations led to the identification of asymptotics of unknown functions of $V$, $\beta$, $U$ metrics (3):

$$V = -r + 2M(u, \theta, \phi) + O(r^{-1}),$$

$$\psi = -c(u,\theta,\phi)c^*(u, \theta, \phi)\frac{1}{4r^2} + O(r^{-4}),$$

$$h_{AB}dx^A dx^B = (d\theta^2 + sin^2\theta d\phi^2) + O(r^{-1}),$$

$$U^A = O(r^{-2}),$$

where $c^*$ is complex-conjugate to $c$.

As a consequence, in the limit at $r \to \infty$ it can be shown that if we take $b = sin^2\theta$, then the metric (3) is transformed to the form of the Minkowski metric in spherical coordinates:

$$lim(ds^2) = du^2 - 2dudr + r^2(d\theta^2 + sin^2\theta d\phi^2). \quad (5)$$

Bondi and Metzner considered the set of all coordinate transformations that preserve the form (3),(4),(5). The result of the work was the construction of a generalized Bondi-Metzner group or Bondi-Metzner-Sachs group.

# 5 Group properties

Group properties of Bondi-Metzner-Sachs group are following.

**Theorem 5.1.** *The supertranslations form a normal Abelian subgroup of the Bondi-Metzner-Sachs group and its quotient group is isomorphic to the homogeneous Lorentz group.*

**Lemma 5.2.** *Translations form a four-dimensional normal subgroup of the Bondi-Metzner-Sachs group*

**Lemma 5.3.** *If the group is a four-dimensional normal subgroup of the Bondi-Metzner-Sachs group, then it is contained in the supertranslation group.*

**Theorem 5.4.** *The only normal Bondi-Metzner-Sachs subgroup is the translation group.*

The proof of the theorems can be found in [2].

# 6 Conclusion

Mail goal of the practice was to get acquainted with different definitions of asymptotically flat spacetimes and asymptotic symmetries. My work is a small introduction in BMS formalism and during studies I found out where this approach can be applied, for example in numerical relativity and special solutions of Einstein's equation.

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# 8 Appendix A[3]

This appendix will provide some explanations to the type of BMS metric (3). The determinant condition requires that $h^{AB}\partial_r h_{AB} = h^{AB}\partial_u h_{AB} = 0$, where $h^{AC}h_{CB} = \delta^A_B$. The covariant derivative $D_A$ of the metric $h_{AB}$ is defined as $D^A = h_{AB}D_B$. Corresponding non-null contravariant components of the metric $g_{\alpha\beta}$:

$$g^{ur} = -e^{-2\beta}, \quad g^{rr} = \frac{V}{r}e^{-2\beta}, \quad g^{rA} = -U^A e^{-2\beta}, \quad g^{AB} = \frac{1}{r^2}h^{AB}$$

We can represent the metric $h_{AB}$ through two functions $\gamma(u,r,\theta,\phi)$ and $\delta(u,r,\theta,\phi)$ which are corresponding to + and $\times$ polarizations of gravitational waves:

$$h_{AB}dx^A dx^B = (e^{2\gamma}d\theta^2 + e^{-2\gamma}sin^2(\theta)d\phi^2)ch(2\delta) + 2sin(\theta)sh(2\delta)d\theta d\phi. \tag{6}$$

This form (6) differs from the one presented in Sachs' original work by one transformation: $\gamma \to \frac{\gamma+\delta}{2}, \quad \delta \to \frac{\gamma-\delta}{2}$.

In Bondi's original paper, an axisymmetric metric was considered, with symmetry with respect to $\phi$, $\delta = U^{\phi} = 0$ and $\gamma = \gamma(u,r,\theta)$, which ultimately gives the Bondi metric:

$$g^{(B)}_{\alpha\beta}dx^{\alpha}dx^{\beta} = (-\frac{V}{r}e^{2\psi} + r^2Ue^{2\gamma})du^2 - 2e^{2\psi}dudr - r^2Ue^{2\gamma}dud\theta + r^2(e^{2\gamma}d\theta^2 + e^{-2\gamma}sin^2(\theta)d\phi^2),$$

where $U = U^{\theta}$, but such a metric is not suitable for describing $\phi$-symmetric rotating bodies.

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