JOINT INSTITUTE FOR NUCLEAR RESEARCH

Bogoliubov Laboratory of Theoretical Physics

FINAL REPORT ON THE START PROGRAMME

STATISTICAL APPROACH FOR HADRON PRODUCTION IN HEAVY ION COLLISION AT HIGH ENERGIES

Supervisor:

Alexandru S. Parvan

Student:

Belyakova Mariya, Moscow, NRNU MEPhI

Participation period:

July 17 – September 3

Dubna, 2022

ABSTRACT

The aim of this work is the study of the statistical hadron model with exactly conserved strange charge of the system. The statistical hadron model is used to describe the yields and ratios of hadrons produced in heavy-ion collisions at high energies. This approach is relevant in the final state of the nuclear collision at chemical freeze-out, when the equilibrium statistical mechanics can be applied. In this case the system of the produced particles can be treated as an ideal gas of different species. Ideal gas following different statistics (Maxwell-Boltzmann, Bose-Einstein and Fermi-Dirac) for canonical and grand canonical ensembles has been considered. The thermodynamic potentials, partition functions and ensemble averages for each statistical ensemble has been calculated. The exact solution for the canonical partition function with exact conservation of net strangeness for ideal gas of different species with quantum and classical statistics of particles has been obtained. The mean occupation numbers has been calculated.

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INTRODUCTION

In heavy ion collisions the strange particle multiplicity has an anomalous rise at the $\sqrt{s}$ ~8 GeV. The observed maximum of the ratio $K^+/_{\pi^+}$ can be considered as the sign of the Quark-gluon plasma (QGP) formation [1, 2, 3]. In order to evaluate the production of different sorts of particles in heavy ion collisions we apply the statistical hadron model with exactly conserved strangeness charge [4]. It also can be used to analyze thermodynamic properties of hadron systems occurring in heavy ion collisions.

The conservation laws of the charges (electric, baryon, strange) are fulfilled in heavy ion collisions. It means that initial net charge of all involved in the collision nucleons should be equal to the net charge of the particles created in considered event. The colliding nucleons do not contain strange quarks, therefore the net strangeness equals zero for all events. On the other hand, nucleons carry baryon and electric charge, hence, these net charges of the system differ from event to event (because collisions have different centrality).

The strange particles produced in heavy ion collisions need to be treated canonically, because the strange charge is conserved exactly and do not differ from event to event. Particles carrying electric or baryon charges need to be treated grand canonically. The net electric and baryon charges are not conserved exactly in the created fireball. The part of the nucleons (the nucleons of spectators) of the colliding heavy ions is not involved in the collision. Therefore, the baryon and electric charges of the system fluctuate from event to event.

In order to consider system strictly, all conserved charges need to be treated canonically. However, in heavy ion collisions electric and baryon charges can be conserved in average, which do not cause sufficient deviations from canonical effects, while particles carrying strange charge cannot be treated grand canonically [4]. In description of heavy ion collisions it is important to impose exact strangeness conservation.

In this work, we consider the statistical hadron model with exactly conserved strange charge of the system [4]. We derive the canonical partition function for the quantum and classical ideal gas of different species of hadrons. The solution of the partition function in the case of the Maxwell-Boltzmann statistics of particles is obtained by two different methods: direct and recurrent equations. For the Bose-Einstein and Fermi-Dirac statistics of particles the exact solutions of the partition functions are obtained by the method of recurrence relation. The mean occupation numbers and other ensemble averages have been obtained.

CHAPTER I. RELATIVISTIC KINIMATICS AND VARIABLES

Let us consider two-body collision. In order to describe the colliding system, we introduce four-momentum for each particle. The components of this four-dimensional vector is following:

$$P^{\mu} = (E,\ p_{x,}\ p_{y,}p_{z,}). \tag{1.1}$$

Relativistic invariant variable s is one of Mandelstam variables and related to four-momentum by following equation [3, 5]:

$$s = (P_1 + P_2)^2 = (P_1 + P_2)_{\mu}(P_1 + P_2)^{\mu}, \tag{1.2}$$

where P_1^{μ}, P_2^{μ} stands for four-momentum of first and second particle respectively. Invariant variable s can be used for finding the relation between energy of the system in one frame of references the the energy of the system in another one.

1.1 LABORATORY FRAME

In laboratory frame of references one of the colliding particle is at rest. This particle is called target (the target rest frame). The second particle is called projectile. In target frame the components of four-momentum have the following values:

$$P_1^{\mu} = (E^{lab}, \vec{p}), \qquad P_2^{\mu} = (m_2, 0), \tag{1.1.1}$$

where $E_1 = E^{lab}$, $E_2 = m_2$, $\overrightarrow{p_1} = \vec{p}$, $\overrightarrow{p_2} = 0$ and $E^{lab} = \sqrt{\vec{p}^2 + {m_1}^2}$. Therefore, we obtain:

$$s = (p_1 + p_2)_{\mu}(p_1 + p_2)^{\mu} = (E^{lab} + m_2)^2 - \vec{p}^2 = {m_1}^2 + {m_2}^2 + 2E^{lab}m_2. \tag{1.1.2}$$

For the particles with equal rest mass m the equation (1.1.2) will simplify to:

$$\sqrt{s} = \sqrt{2m(E^{lab} + m)}\ . \tag{1.1.4}$$

For $E^{lab} \gg m$ (corresponds to ultra-relativistic case), we have

$$\sqrt{s} \approx \sqrt{2E^{lab}m}\ . \tag{1.1.4a}$$

The Lorenz invariant $\sqrt{s}$ describes the energy of the system that occurs in relativistic collisions. And $\sqrt{s}$ can be obtained when the E^{lab} is known. But in the real experiment for the heavy ion collision this quantity E^{lab} can't be defined directly. We do know the energy of one proton accelerated by strong electric or magnetic field. By knowing how many protons ion is comprised of, we can define the net energy E^{lab}.

Let's consider heavy-ion collisions. For that we consider the invariant quantity $\sqrt{s_{NN}}$ defined as

$$\sqrt{s_{NN}} = \sqrt{\left(\frac{P_1}{A_1} + \frac{P_2}{A_2}\right)^2}\ , \tag{1.1.5}$$

where A_1 and A_2 are the number of nucleons of colliding ions.

After simplifying (1.1.5), we obtain:

$$\sqrt{s_{NN}} = \sqrt{\frac{m_1^2}{A_1^2} + \frac{m_2^2}{A_2^2} + \frac{2E^{lab}m_1}{A_1A_2}} \ . \tag{1.1.5a}$$

When $m_1 = m_2 = m$ $(A_1 = A_2 = A)$, we obtain:

$$\sqrt{s_{NN}} = \frac{1}{A}\sqrt{2m(E^{lab} + m)} \ . \tag{1.1.5b}$$

For $E^{lab} \gg m$, we have

$$\sqrt{s_{NN}} \approx \frac{\sqrt{2E^{lab}m}}{A} \ . \tag{1.1.5c}$$

The quantity $\sqrt{s_{NN}}$ reflects the energy of the matter that occur in heavy ion collision on a couple of colliding nucleons.

1.2 CENTER-OF-MASS FRAME

In center-of-mass frame the momenta of two colliding particles have equal magnitude but opposite directions, hence net momentum equals zero [3, 5]. The four-momentum of colliding particles is defined as

$$p_1^{\mu} = (E_1^c, \vec{p}), \tag{1.2.1}$$

$$p_2^{\mu} = (E_2^c, -\vec{p}). \tag{1.2.2}$$

Therefore, the $\sqrt{s}$ can be written as

$$\sqrt{s} = \sqrt{(E_1^c + E_2^c)^2} = E_1^c + E_2^c. \tag{1.2.3}$$

When $m_1 = m_2$, E_1^c and E_2^c also supposed to be equal (derives from the definition of the energy in relativistic case). Therefore, (1.2.3) can be simplified as

$$\sqrt{s} = 2E^c, \tag{1.2.3a}$$

where E^c is the energy of each particle with equal masses in center-of-mass frame.

In the case of center-of-mass frame the $\sqrt{s_{NN}}$ can be obtain from equation (1.1.5) :

$$\sqrt{s_{NN}} = \sqrt{\frac{m_1^2}{A_1^2} + \frac{m_2^2}{A_2^2} - \frac{m_1^2+m_2^2}{A_1A_2} + \frac{(E_1^c+E_2^c)^2}{A_1A_2}} \ . \tag{1.2.4}$$

For $m_1 = m_2 = m$ and $A_1 = A_2 = A$ we have:

$$\sqrt{s_{NN}} = \frac{2E^c}{A}, \tag{1.2.4a}$$

where A corresponds to the mass number of equal colliding ions.

1.3 HEAD-ON COLLISIONS

Let’s consider frame of references in which particles approach each other along one line- head-on frame. Four-momentum of each particle has following components:

$$p_1^{\mu} = (E_1^h, p_1\overrightarrow{e_p}), \tag{1.3.1}$$

$$p_2^{\mu} = (E_2^h, -p_2\overrightarrow{e_p}), \tag{1.3.2}$$

where $\overrightarrow{e_p}$ is unite vector defining direction of approaching, $p_1 = |\overrightarrow{p_1}|$ and $p_2 = |\overrightarrow{p_2}|$. In head-on frame scalar variable $\sqrt{s}$ is obtained by the equation:

$$\sqrt{s} = \sqrt{(E_1^h + E_2^h)^2 - (p_1\overrightarrow{e_p} - p_2\overrightarrow{e_p})^2} = \sqrt{{m_1}^2 + {m_2}^2 + 2E_1^h E_2^h + 2p_1 p_2}\ . \tag{1.3.3}$$

For the ultra-relativistic case ($E_{1,2}^h \gg m_{1,2}$) equation simplifies:

$$\sqrt{s} = 2\sqrt{E_1^h E_2^h}\ . \tag{1.3.4}$$

We obtain

$$\sqrt{s_{NN}} = \sqrt{\left(\frac{m_1}{A_1}\right)^2 + \left(\frac{m_2}{A_2}\right)^2 + 2\,\frac{E_1^h E_2^h + p_1 p_2}{A_1 A_2}}\ . \tag{1.3.5}$$

When the rest masses of colliding particles are equal, hence $A_1 = A_2 = A$, we rewrite equation (1.3.5):

$$\sqrt{s_{NN}} = \frac{1}{A}\sqrt{2\left(m^2 + E_1^h E_2^h + p_1 p_2\right)}. \tag{1.3.5a}$$

For $E_1^h, E_1^h \gg m$, we have

$$\sqrt{s_{NN}} = \frac{2\sqrt{E_1^h E_2^h}}{A}\ . \tag{1.3.5b}$$

CHAPTER II. EQUILIBRIUM TERMODYNAMICS

Thermodynamic equilibrium is the state of the system, when its macroscopic parameters do not change (under the condition that system is isolated). It is obvious that little fluctuations of variables of state take place due to thermal motion of the particles [6].

All thermodynamic quantities in the equilibrium thermodynamics can be divided into two groups: extensive and intensive. When we divide system into two, extensive variables satisfies relation [7]:

$$F^{1+2} = F^1 + F^2, \tag{2.1}$$

while intensive variables do not change:

$$f^{1+2} = f^1 = f^2. \tag{2.2}$$

The thermodynamic system is comprehensively described by its particular thermodynamic potential. Thermodynamic potential is the function $Y = Y(x_1, x_2, \dots, x_n)$, where $x_1, x_2, \dots, x_n$ are independent variables called variables of state. These variables fix macroscopic state of the system. The partial derivatives of the thermodynamic potential: $\frac{\partial Y}{\partial x_i}$ - define the thermodynamic quantities of the system at the equilibrium.

All thermodynamic quantities are homogeneous functions of degree k=1 (extensive) or k=0 (intensive) [8, 9]:

$$A(\lambda x_1, \lambda x_2, \dots, \lambda x_n, x_{n+1}, \dots, x_m) = \lambda^k A(x_1, x_2, \dots, x_n, x_{n+1}, \dots, x_m), \tag{2.3}$$

where x_i is extensive variable for *i=1,...,n* and intensive for *i=n+1,...,m.*

2.1 FUNDAMENTAL THERMODYNAMIC POTENTIAL

The main quantity of the equilibrium thermodynamics is fundamental thermodynamic potential (energy) [7], which is a function of three independent variables $E = E(S, V, N)$, where S is the entropy of the system, V is the volume occupied by the system and N is the number of particles that comprise this system. The first differential of the fundamental thermodynamic potential can be written as [10]

$$dE = \frac{\partial E}{\partial S} dS + \frac{\partial E}{\partial V} dV + \frac{\partial E}{\partial N} dN = TdS - pdV + \mu dN, \tag{2.1.1}$$

where

$$T \equiv \frac{\partial E}{\partial S}, \tag{2.1.2}$$

$$p \equiv -\frac{\partial E}{\partial V}, \tag{2.1.3}$$

$$\mu \equiv \frac{\partial E}{\partial N}. \tag{2.1.4}$$

The fundamental thermodynamic potential defines the thermodynamic quantities: pressure p, temperature T and chemical potential μ. The equation (2.1.1) is the fundamental equation of thermodynamics, which combine first and second laws of thermodynamics.

In this ensemble variables of state satisfies relations of homogeneity [7, 10]:

$$E(\lambda S, \lambda V, \lambda N) = \lambda E(S, V, N), \tag{2.1.5}$$
$$T(\lambda S, \lambda V, \lambda N) = T(S, V, N), \tag{2.1.6}$$
$$\mu(\lambda S, \lambda V, \lambda N) = \mu(S, V, N), \tag{2.1.7}$$
$$p(\lambda S, \lambda V, \lambda N) = p(S, V, N), \tag{2.2.8}$$

where *E* is extensive variable, while *T*, *p* and μ are intensive.

2.2 THERMODYNAMIC POTENTIAL OF THE CLOSED SYSTEM

The independent variables of state in the fundamental thermodynamic potential are (S, N, V). The Helmholtz thermodynamic potential F is obtained from fundamental thermodynamic potential by changing the variables S and T and by the Legendre transform [11] (see Appendix B):

$$F(T, V, N) = E(S(T, V, N) - TS(T, V, N), \tag{2.2.1}$$

where S is obtained from the equation: $\frac{\partial E}{\partial S} = T$. In new set of independent variables the first differetial is defined in the following way [10]:

$$dF = \frac{\partial F}{\partial T} dT + \frac{\partial F}{\partial V} dV + \frac{\partial F}{\partial N} dN = -SdT - pdV + \mu dN, \tag{2.2.2}$$

where

$$\frac{\partial F}{\partial T} = \frac{\partial E}{\partial T} - \frac{\partial (ST)}{\partial T} = \frac{\partial E}{\partial S}\frac{\partial S}{\partial T} - S - T\frac{\partial S}{\partial T} = -S, \qquad \frac{\partial F}{\partial V} = \frac{\partial E}{\partial V} = -p, \qquad \frac{\partial F}{\partial N} = \frac{\partial E}{\partial N} = \mu. \tag{2.2.3}$$

For the Helmholtz thermodynamic potential and its variables we can write equations [9, 10']:

$$F(T, \lambda V, \lambda N) = \lambda F(T, V, N), \tag{2.2.4}$$
$$S(T, \lambda V, \lambda N) = S(T, V, N), \tag{2.2.5}$$
$$\mu(T, \lambda V, \lambda N) = \mu(T, V, N), \tag{2.2.6}$$
$$p(T, \lambda V, \lambda N) = p(T, V, N). \tag{2.2.7}$$

The Helmholtz thermodynamic potential defines the canonical ensemble, which is fixed by the variables of state (*T*, *V*, *N*).

2.3 THERMODYNAMIC POTENTIAL OF THE ISOLATED SYSTEM

The thermodynamic potential of isolated system is entropy as the function of the independent variables of state (E, V, N) [7]

$$S = S(E, V, N)\ . \tag{2.3.1}$$

Then first deferential of entropy S can be written as

$$dS = \frac{\partial S}{\partial E} dE + \frac{\partial S}{\partial V} dV + \frac{\partial S}{\partial N} dN. \tag{2.3.2}$$

Let us define partial derivatives in (2.3.2). For that we divide equation (2.1.1) on $\frac{\partial E}{\partial S} = T$, then we obtain:

$$\frac{1}{T}dE = dS - \frac{p}{T}dV + \frac{\mu}{T}dN\ . \qquad (2.3.3)$$

Therefore, we can exert dS [10]:

$$dS = \frac{1}{T}dE + \frac{p}{T}dV - \frac{\mu}{T}dN. \qquad (2.3.4)$$

Eventually, we obtain [10]:

$$\frac{\partial S}{\partial E} = \frac{1}{\frac{\partial E}{\partial S}} = \frac{1}{T}\,, \qquad (2.3.5)$$

$$\frac{\partial S}{\partial V} = -\frac{1}{T}\frac{\partial E}{\partial V} = \frac{p}{T}\,, \qquad (2.3.6)$$

$$\frac{\partial S}{\partial N} = -\frac{1}{T}\frac{\partial E}{\partial N} = -\frac{\mu}{T}\,. \qquad (2.3.7)$$

In this ensemble variables of state satisfies following relations [7, 10]:

$$S(\lambda E, \lambda V, \lambda N) = \lambda S(E, V, N), \qquad (2.3.8)$$
$$T(\lambda E, \lambda V, \lambda N) = T(E, V, N), \qquad (2.3.9)$$
$$\mu(\lambda E, \lambda V, \lambda N) = \mu(E, V, N), \qquad (2.3.10)$$
$$p(\lambda E, \lambda V, \lambda N) = p(E, V, N). \qquad (2.3.11)$$

2.4 THERMODYNAMIC POTENTIAL OF THE OPEN SYSTEM

The thermodynamic potential of the open system is grand thermodynamic potential Ω as the function of the variables of state (T, V, μ). It can be obtained from the fundamental thermodynamic potential by changing variables of state (S, V, N) to (T, V, μ) and the Legendre transform [11]:

$$\Omega(T, V, \mu) = E\big(S(T, V, \mu), V, N(T, V, \mu)\big) - TS(T, V, \mu) - \mu N(T, V, \mu), \qquad (2.4.1)$$

where S and N are obtained from the following equations respectively:

$$\frac{\partial E}{\partial S} = T, \quad \frac{\partial E}{\partial N} = \mu. \qquad (2.4.2)$$

The first differential is defined in the following way [10]:

$$d\Omega = \frac{\partial \Omega}{\partial T}dT + \frac{\partial \Omega}{\partial V}dV + \frac{\partial \Omega}{\partial \mu}d\mu, \qquad (2.4.3)$$

where

$$\frac{\partial \Omega}{\partial T} = \frac{\partial E}{\partial T} - \frac{\partial (ST)}{\partial T} - \frac{\partial (\mu N)}{\partial T} = \frac{\partial E}{\partial S}\frac{\partial S}{\partial T} + \frac{\partial E}{\partial N}\frac{\partial N}{\partial T} - S - T\frac{\partial S}{\partial T} - \mu\frac{\partial N}{\partial T} = -S, \qquad (2.4.4)$$

$$\frac{\partial \Omega}{\partial V} = \frac{\partial E}{\partial V} - \frac{\partial (ST)}{\partial V} - \frac{\partial (\mu N)}{\partial V} = \frac{\partial E}{\partial V} + \frac{\partial E}{\partial S}\frac{\partial S}{\partial V} + \frac{\partial E}{\partial N}\frac{\partial N}{\partial V} - T\frac{\partial S}{\partial V} - \mu\frac{\partial N}{\partial V} = -p, \qquad (2.4.5)$$

$$\frac{\partial \Omega}{\partial \mu} = \frac{\partial E}{\partial \mu} - \frac{\partial (ST)}{\partial \mu} - \frac{\partial (\mu N)}{\partial \mu} = \frac{\partial E}{\partial \mu} + \frac{\partial E}{\partial S}\frac{\partial S}{\partial \mu} + \frac{\partial E}{\partial N}\frac{\partial N}{\partial \mu} - T\frac{\partial S}{\partial \mu} - \mu\frac{\partial N}{\partial \mu} - N = -N. \qquad (2.4.6)$$

As it is done above, let us write relation for extensive and intensive variables [7, 10]:

$$\Omega(T, \lambda V, \mu) = \lambda\Omega(T, V, \mu), \tag{2.4.7}$$

$$S(T, \lambda V, \mu) = T(T, V, \mu), \tag{2.4.8}$$

$$N(T, \lambda V, \mu) = N(T, V, \mu), \tag{2.4.9}$$

$$p(T, \lambda V, \mu) = p(T, V, \mu). \tag{2.4.10}$$

CHAPTER III. EQUILIBRIUM STATISTICAL MECHANICS

In equilibrium thermodynamics macroscopic variables define the system comprehensively. In statistical mechanics we need to introduce more parameters of the system: microstates and its probabilities.

The system with fixed variables of sate (that define macrostate) in statistical approach can be obtained by different microstates. Every of them is comprised of different combinations of single-particle states and has different probabilities to occur.

The first step in statistical approach is to define probabilities of microstates. Let us consider the Boltzmann-Gibbs entropy S [7, 9] as

$$S = -\sum_i p_i \ln p_i \,, \tag{3.1}$$

where p_i is the probability of i-th microstate of the system. The statistical averages are defined as:

$$\langle A \rangle = \sum_i A_i p_i \,, \tag{3.2}$$

where A_i is the value of quantity A in i-th microstate. The norm equation for the probabilities of microstates is

$$\sum_i p_i = 1. \tag{3.3}$$

Let us consider the thermodynamic system defined by the variables of state $(X^1, \dots, X^n)$ and thermodynamic potential $\mathrm{Y}(X^1, \dots, X^n)$. For i-th microstate thermodynamic potential has the value [7, 9]:

$$Y_i = Y_i(X_i^1, X_i^2, \dots, X_i^n). \tag{3.4}$$

All thermodynamic potentials considered above are extensive functions (chapter 2), therefore we can find their averages over all microstate using equation (3.2) [9]:

$$Y(X^1, \dots, X^n) = \sum_i Y_i p_i \,. \tag{3.5}$$

Unknown set of probabilities $\{p_i\}$ we find using relation (3.3) and the condition of extremum of thermodynamic potential as the function of $\{p_i\}$. In order to define $\{p_i\}$, the method of Lagrange can be applied [9]:

$$\begin{cases} \Phi = Y(p_1, p_2, p_3, \dots, p_W, X^1, \dots, X^n) - \lambda(\sum_i p_i - 1), \\ \dfrac{\partial \Phi}{\partial p_i} = 0, \quad i = 1, 2, \dots, W. \end{cases} \tag{3.6}$$

We obtain [9]

$$Y + p_i \frac{\partial Y}{\partial p_i} - \lambda = 0. \tag{3.7}$$

From this equation we obtain $p_i = p_i(\lambda, X^1, \dots, X^n)$, where λ is unknown Lagrange parameter. Substituting p_i into equation (3.3) we obtain $\lambda(X^1, \dots, X^n)$.

3.1 CANONICAL ENSEMBLE

3.1.1 *General theory*

Let us consider canonical ensemble in statistical approach which can be described with the use of Helmholtz thermodynamic potential $F = F(\{p_i\}, T, V, N)$ [7], where p_i is the probability of corresponding microsate, T, V, N are variables of state that fix macroscopic state. Using (3.5) and (2.2.1), we obtain [12]

$$F = \sum_i p_i F_i = \sum_i p_i (E_i + T\ln p_i), \quad F_i = E_i + T\ln p_i. \tag{3.1.1}$$

Applying the method of Lagrange multiplier [9], we have

$$\begin{cases} \Phi = F(p_1, p_2, p_3, \ldots, p_W, T, V, N) - \lambda(\sum_i p_i - 1), \\ \frac{\partial \Phi}{\partial p_i} = 0, i = 1, 2, \ldots, W, \end{cases} \tag{3.1.2}$$

where W is the number of microstates. From these equations we obtain [9]

$$F_i + p_i \frac{\partial F_i}{\partial p_i} - \lambda = 0. \tag{3.1.3}$$

Substituting (3.1.1) into (3.1.3), we have

$$E_i + T\ln p_i + p_i \frac{T}{p_i} + \lambda = 0. \tag{3.1.3a}$$

Then we obtain the probability of *i-th* microstate:

$$p_i = e^{\frac{1}{T}(\lambda - E_i - T)}. \tag{3.1.4}$$

Substituting (3.1.4) into (3.3), we obtain

$$\lambda = T - T\ln Z, \qquad Z = \sum_i e^{-\frac{1}{T}E_i}, \tag{3.1.5}$$

where Z is partition function. Substituting (3.1.5) in (3.1.4):

$$p_i = \frac{1}{Z} e^{-\frac{1}{T}E_i}. \tag{3.1.4a}$$

Now we can write *S*, *E* and the Helmholtz thermodynamic potential *F* as

$$S = \ln Z + \frac{E}{T}, \tag{3.1.6}$$

$$E = \frac{1}{Z} \sum_i E_i e^{-\frac{1}{T}E_i}, \tag{3.1.7}$$

$$F = E - TS = -T\ln Z. \tag{3.1.8}$$

3.1.2 *Ideal gas*

For ideal gas energy and number of particles of the microstate can be represented as sum over single-particle states [6, 7]:

$$E = \sum_{\vec{p}} n_{\vec{p}}\, \varepsilon_{\vec{p}}, \tag{3.1.9}$$

$$N = \sum_{\vec{p}} n_{\vec{p}}, \tag{3.1.10}$$

where $n_{\vec{p}}$ is occupation number of $\vec{p}$-th single-particle state.

The statistical averages can be written as [6, 12]

$$\langle A\rangle = \frac{1}{Z}\sum_{\{n_{\vec{p}}\}} A(\{n_{\vec{p}}\})e^{-\frac{1}{T}\sum_{\vec{p}} n_{\vec{p}}\varepsilon_{\vec{p}}}\,\delta\left(\sum_{\vec{p}} n_{\vec{p}} - N\right), \tag{3.1.11}$$

$$Z = \sum_{\{n_{\vec{p}}\}} G(\{n_{\vec{p}}\})e^{-\frac{1}{T}\sum_{\vec{p}} n_{\vec{p}}\varepsilon_{\vec{p}}}\,\delta\left(\sum_{\vec{p}} n_{\vec{p}} - N\right), \tag{3.1.12}$$

where $G(\{n_{\vec{p}}\})$=1 for Bose-Einstein and Fermi-Dirac and $G(\{n_{\vec{p}}\}) = \frac{1}{\prod_{\vec{p}} n_{\vec{p}}!}$ for Maxwell-Boltzmann statistics of particles. The occupation numbers $n_{\vec{p}} = 0,1,\ldots$ for Bose-Einstein and Maxwell-Boltzmann statistics and $n_{\vec{p}} = 0,1$ for Fermi-Dirac statistics. In partition function summation is taken over all microstates (every microstate is the combination of single-particle states). Summation is written in terms of occupation numbers [7], which reflects number of particle in every single-particle state: $\sum_{\{n_\alpha\}} = \sum_{n_1}\sum_{n_2}\sum_{n_3}\ldots.$ is product of sum over number of particles in every single-particle state.

Let us derive partition function for ideal gas of N particles for all types of single-particle statistics: Maxwell-Boltzmann, Bose-Einstein and Fermi-Dirac.

Classical statistics. First of all, let us consider ideal gas of classical particles. The partition function will have the following form [7, 12] (see Appendix C):

$$Z_N = \frac{1}{N!}\sum_{\{n_{\vec{p}}\}}\frac{N!}{\prod_{\vec{p}} n_{\vec{p}}!}e^{-\frac{1}{T}\sum_{\vec{p}} n_{\vec{p}}\varepsilon_{\vec{p}}}\,\delta\left(\sum_{\vec{p}} n_{\vec{p}} - N\right) = \frac{1}{N!}\left(\sum_{\vec{p}} e^{-\frac{1}{T}\varepsilon_{\vec{p}}}\right)^N. \tag{3.1.13}$$

The mean occupation numbers can written as [12]

$$\langle n_{\vec{p}\prime}\rangle = \frac{1}{Z}\sum_{\{n_{\vec{p}}\}}\frac{1}{\prod_{\vec{p}} n_{\vec{p}}!}n_{\vec{p}\prime}e^{-\frac{1}{T}\sum_{\vec{p}} n_{\vec{p}}\varepsilon_{\vec{p}}}\,\delta\left(\sum_{\vec{p}} n_{\vec{p}} - N\right)$$

$$= -T\frac{1}{Z}\frac{\partial}{\partial\varepsilon_{\vec{p}\prime}}\sum_{\{n_{\vec{p}}\}}\frac{1}{\prod_{\vec{p}} n_{\vec{p}}!}e^{-\frac{1}{T}\sum_{\vec{p}} n_{\vec{p}}\varepsilon_{\vec{p}}}\,\delta\left(\sum_{\vec{p}} n_{\vec{p}} - N\right)$$

$$= \frac{1}{Z}\frac{N}{N!}\left(\sum_{\vec{p}} e^{-\frac{1}{T}\varepsilon_{\vec{p}}}\right)^{N-1}e^{-\frac{1}{T}\varepsilon_{\vec{p}\prime}}. \tag{3.1.14}$$

Substituting (3.1.13) in (3.1.14), we obtain:

$$\langle n_{\vec{p}\prime}\rangle = \frac{N}{\sum_{\vec{p}} e^{-\frac{1}{T}\varepsilon_{\vec{p}}}}e^{-\frac{1}{T}\varepsilon_{\vec{p}\prime}} = \frac{N}{Z_1}e^{-\frac{1}{T}\varepsilon_{\vec{p}\prime}}, \tag{3.1.15}$$

where Z_1=$\sum_{\vec{p}} e^{-\frac{1}{T}\varepsilon_{\vec{p}}}$ is partition function of the system comprised of one particle. We have obtained partition function and average occupation numbers for particles following Maxwell-Boltzmann statistics.

Quantum statistics. Let us now consider the ideal gas following Bose-Einstein statistics. The partition function in the representation of occupation numbers has form [12]:

$$Z_N = \sum_{\{n_{\vec{p}}\}} e^{-\frac{1}{T}\sum_{\vec{p}}\varepsilon_{\vec{p}} n_{\vec{p}}}\,\delta\left(\sum_{\vec{p}} n_{\vec{p}} - N\right). \tag{3.1.16}$$

The partition function can be solved by recurrence equations [12-16] :

$$Z_N = \sum_{l=1}^{N}\omega_l Z_{N-l}, \qquad Z_0 = 1, \tag{3.1.17}$$

where $\omega_l = \sum_{\vec{p}} e^{-\frac{1}{T}\varepsilon_{\vec{p}} l}$.

For proving relation (3.1.17) we use the equation [12]:

$$\sum_{l=1}^{N} x^l \sum_{n_{\vec{p}'}=0}^{N-l} x_{p'}{}^{n_{\vec{p}'}} \delta(\sum_{\vec{p}\neq\vec{p}'} n_{\vec{p}} - (N - l - n_{\vec{p}'}))$$

$$= \sum_{n_{\vec{p}'}=0}^{N} n_{\vec{p}'} x_{p'}{}^{n_{\vec{p}'}} \delta(\sum_{\vec{p}\neq\vec{p}'} n_{\vec{p}} - (N - n_{\vec{p}'})) \ . \qquad (3.1.18)$$

By replacing variables and changing $n_{\vec{p}} \to n_{\vec{p}} + l$, we obtain

$$\sum_{l=1}^{N} x^l \sum_{n_{\vec{p}'}=0}^{N-l} x_{p'}{}^{n_{\vec{p}'}} \delta(\sum_{\vec{p}\neq\vec{p}'} n_{\vec{p}} - (N - l - n_{\vec{p}'}))$$

$$= \sum_{l=1}^{N} \sum_{n_{\vec{p}'}=0}^{N-l} x_{p'}{}^{n_{\vec{p}'}+l} \delta(\sum_{\vec{p}\neq\vec{p}'} n_{\vec{p}} - (N - l - n_{\vec{p}'}))$$

$$= \sum_{l=1}^{N} \sum_{n_{\vec{p}'}=l}^{N} x_{p'}{}^{n_{\vec{p}'}} \delta(\sum_{\vec{p}\neq\vec{p}'} n_{\vec{p}} - (N - n_{\vec{p}'})). \qquad (3.1.19)$$

Then we introduce the step function $\Theta(x) = \begin{cases} 1, x \geq 0 \\ 0, x < 0 \end{cases}$. Function $\Theta\left(n_{\vec{p}'} - l\right)$ gives us an opportunity to start summation over $n_{\vec{p}'}$ from zero, because for every $n_{\vec{p}'} < l$ function $\Theta\left(n_{\vec{p}'} - l\right) = 0$:

$$\sum_{l=1}^{N} \sum_{n_{\vec{p}'}=l}^{N} x_{p'}{}^{n_{\vec{p}'}} \delta(\sum_{\vec{p}\neq\vec{p}'} n_{\vec{p}} - (N - n_{\vec{p}'}))$$

$$= \sum_{l=1}^{N} \sum_{n_{\vec{p}'}=0}^{N} \Theta\left(n_{\vec{p}'} - l\right) x_{p'}{}^{n_{\vec{p}'}} \delta(\sum_{\vec{p}\neq\vec{p}'} n_{\vec{p}} - (N - n_{\vec{p}'})). \qquad (3.1.20)$$

Substituting right-hand side of the relation $\sum_{l=1}^{N} \Theta\left(n_{\vec{p}'} - l\right) = n_{\vec{p}'}$ into (3.1.20), we obtain:

$$\sum_{l=1}^{N} x^l \sum_{n_{\vec{p}'}=0}^{N-l} x_{p'}{}^{n_{\vec{p}'}} \delta(\sum_{\vec{p}\neq\vec{p}'} n_{\vec{p}} - (N - l - n_{\vec{p}'}))$$

$$= \sum_{n_{\vec{p}'}=0}^{N} n_{\vec{p}'} x_{p'}{}^{n_{\vec{p}'}} \delta(\sum_{\vec{p}\neq\vec{p}'} n_{\vec{p}} - (N - n_{\vec{p}'})) \ .$$

Now we can get back to the equation (3.1.17). For every microstate we can write an equation $\sum_{\vec{p}} n_{\vec{p}} = N$. Substituting it into (3.1.16), we obtain

$$Z_N = \frac{1}{N} \sum_{\{n_{\vec{p}}\}} e^{-\frac{1}{T}\sum_{\vec{p}} \varepsilon_{\vec{p}} n_{\vec{p}}} \delta(\sum_{\vec{p}} n_{\vec{p}} - N) \times \sum_{\vec{p}'} n_{\vec{p}'}$$

$$= \frac{1}{N} \sum_{\vec{p}'} \sum_{\{n_{\vec{p}}\}_{\vec{p}\neq\vec{p}'}} e^{-\frac{1}{T}\sum_{\vec{p}\neq\vec{p}'} \varepsilon_{\vec{p}} n_{\vec{p}}} \sum_{n_{\vec{p}'}} n_{\vec{p}'} e^{-\frac{1}{T}\varepsilon_{\vec{p}'} n_{\vec{p}'}} \delta\left(\sum_{\vec{p}\neq\vec{p}'} n_{\vec{p}} - \left(N - n_{\vec{p}'}\right)\right). \qquad (3.1.21)$$

Substituting left-hand side of equation (3.1.18) into (3.1.21) with $x_{p'} = e^{-\frac{1}{T}\varepsilon_{\vec{p}'}}$, we obtain

$$Z_N = \frac{1}{N} \sum_{\vec{p}'} \sum_{\{n_{\vec{p}}\}_{\vec{p}\neq\vec{p}'}} e^{-\frac{1}{T}\sum_{\vec{p}\neq\vec{p}'} \varepsilon_{\vec{p}} n_{\vec{p}}} \sum_{l=1}^{N} e^{-\frac{1}{T}\varepsilon_{\vec{p}'} l} \sum_{n_{\vec{p}'}=0}^{N-l} e^{-\frac{1}{T}\varepsilon_{\vec{p}'} n_{\vec{p}'}} \delta(\sum_{\vec{p}\neq\vec{p}'} n_{\vec{p}} - (N - l - n_{\vec{p}'}))$$

$$= \frac{1}{N} \sum_{l=1}^{N} \sum_{\vec{p}'} e^{-\frac{1}{T}\varepsilon_{\vec{p}'} l} \sum_{\{n_{\vec{p}}\}} e^{-\frac{1}{T}\sum_{\vec{p}} \varepsilon_{\vec{p}} n_{\vec{p}}} \delta(\sum_{\vec{p}} n_{\vec{p}} - (N - l)) = \frac{1}{N} \sum_{l=1}^{N} \left[\sum_{\vec{p}'} e^{-\frac{1}{T}\varepsilon_{\vec{p}'} l}\right] Z_{N-l} \ . \qquad (3.1.22)$$

For the mean occupation numbers we can write [12]:

$$\langle n_{\vec{p}'}\rangle = \frac{1}{Z_N}\sum_{\{n_{\vec{p}}\}} n_{\vec{p}'} e^{-\frac{1}{T}\sum_{\vec{p}}\varepsilon_{\vec{p}} n_{\vec{p}}}\,\delta\left(\sum_{\vec{p}} n_{\vec{p}} - N\right)$$

$$\sum_{\vec{p}}\frac{1}{Z_N}\sum_{\{n_{\vec{p}}\}_{\vec{p}\neq\vec{p}'}} e^{-\frac{1}{T}\sum_{\vec{p}\neq\vec{p}'}\varepsilon_{\vec{p}} n_{\vec{p}}}\sum_{n_{\vec{p}'}} n_{\vec{p}'}\, e^{-\frac{1}{T}\varepsilon_{\vec{p}'} n_{\vec{p}'}}\delta\left(\sum_{\vec{p}\neq\vec{p}'} n_{\vec{p}} - (N - n_{\vec{p}'}\,)\right)$$

$$= \frac{1}{Z_N}\sum_{l=1}^{N} e^{-\frac{1}{T}\varepsilon_{\vec{p}'} l} Z_{N-l}\,. \tag{3.1.23}$$

Let us consider Fermi-Dirac statistic. We need to prove the relation [12]:

$$\sum_{l=1}^{N} x^l (-1)^{l+1}\sum_{n_{\vec{p}'=0}}^{min\,[N-l,1]} x^{n_{\vec{p}'}}\delta\left(\sum_{\vec{p}\neq\vec{p}'} n_{\vec{p}} - (N - l - n_{\vec{p}'})\right)$$

$$= \sum_{n_{\vec{p}'=0}}^{1} n_{\vec{p}'} x^{n_{\vec{p}'}}\delta\left(\sum_{\vec{p}\neq\vec{p}'} n_{\vec{p}} - (N - n_{\vec{p}'})\right). \tag{3.1.24}$$

Writing all components of left-hand side sum, we obtain

$$\sum_{l=1}^{N} x^l (-1)^{l+1}\sum_{n_{\vec{p}'=0}}^{min\,[N-l,1]} x^{n_{\vec{p}'}}\delta\left(\sum_{\vec{p}\neq\vec{p}'} n_{\vec{p}} - (N - l - n_{\vec{p}'})\right)$$

$$= x\left(\delta\left(\sum_{\vec{p}\neq\vec{p}'} n_{\vec{p}} - (N-1)\right) + x\,\delta\left(\sum_{\vec{p}\neq\vec{p}'} n_{\vec{p}} - (N-2)\right)\right)$$

$$-x^2\delta\left(\left(\sum_{\vec{p}\neq\vec{p}'} n_{\vec{p}} - (N-2)\right) + x\delta\left(\sum_{\vec{p}\neq\vec{p}'} n_{\vec{p}} - (N-2)\right)\right) + \cdots = x\left(\delta\left(\sum_{\vec{p}\neq\vec{p}'} n_{\vec{p}} - (N-1)\right)\right). \tag{3.1.25}$$

Right-hand side of equation (3.1.25) can be written in following form:

$$x\left(\delta\left(\sum_{\vec{p}\neq\vec{p}'} n_{\vec{p}} - (N-1)\right)\right) = \sum_{n_{\vec{p}'=0}}^{1} n_{\vec{p}'} x^{n_{\vec{p}'}}\delta\left(\sum_{\vec{p}\neq\vec{p}'} n_{\vec{p}} - (N - n_{\vec{p}'})\right). \tag{3.1.26}$$

Let us derive recurrence representation of partition function Z_N :

$$Z_N = \frac{1}{N}\sum_{\{n_{\vec{p}}\}} e^{-\frac{1}{T}\sum_{\vec{p}}\varepsilon_{\vec{p}} n_{\vec{p}}}\,\delta\left(\sum_{\vec{p}} n_{\vec{p}} - N\right)\times\sum_{\vec{p}'} n_{\vec{p}'} =$$

$$= \frac{1}{N}\sum_{\vec{p}'}\sum_{\{n_{\vec{p}}\}_{\vec{p}\neq\vec{p}'}} e^{-\frac{1}{T}\sum_{\vec{p}\neq\vec{p}'}\varepsilon_{\vec{p}} n_{\vec{p}}}\sum_{n_{\vec{p}'}} n_{\vec{p}'} e^{-\frac{1}{T}\varepsilon_{\vec{p}'} n_{\vec{p}'}}\delta\left(\sum_{\vec{p}\neq\vec{p}'} n_{\vec{p}} - \left(N - n_{\vec{p}'}\right)\right). \tag{3.1.27}$$

Substituting (3.1.24) in (3.1.27), we obtain

$$Z_N = \frac{1}{N}\sum_{\vec{p}'}\sum_{\{n_{\vec{p}}\}_{\vec{p}\neq\vec{p}'}} e^{-\frac{1}{T}\sum_{\vec{p}\neq\vec{p}'}\varepsilon_{\vec{p}} n_{\vec{p}}}\sum_{l=1}^{N} e^{-\frac{1}{T}\varepsilon_{\vec{p}'} l}(-1)^{l+1}\times$$

$$\times\sum_{n_{\vec{p}'=0}}^{min\,[N-l,1]} e^{-\frac{1}{T}\varepsilon_{\vec{p}'} n_{\vec{p}'}}\delta\left(\sum_{\vec{p}\neq\vec{p}'} n_{\vec{p}} - (N - l - n_{\vec{p}'})\right) = \frac{1}{N}\sum_{l=1}^{N}(-1)^{l+1}\omega_l Z_{N-l}\,. \tag{3.1.28}$$

For occupation numbers [12]:

$$\langle n_{\vec{p}'}\rangle = \frac{1}{Z_N}\sum_{\{n_{\vec{p}}\}} n_{\vec{p}'} e^{-\frac{1}{T}\sum_{\vec{p}}\varepsilon_{\vec{p}} n_{\vec{p}}}\,\delta\left(\sum_{\vec{p}} n_{\vec{p}} - N\right)$$

$$= \frac{1}{Z_N}\sum_{\{n_{\vec{p}}\}_{\vec{p}\neq\vec{p}'}} e^{-\frac{1}{T}\sum_{\vec{p}\neq\vec{p}'}\varepsilon_{\vec{p}} n_{\vec{p}}} \sum_{n_{\vec{p}'}} n_{\vec{p}'}\, e^{-\frac{1}{T}\varepsilon_{\vec{p}'} n_{\vec{p}'}}\,\delta\left(\sum_{\vec{p}\neq\vec{p}'} n_{\vec{p}} - \left(N - n_{\vec{p}'}\right)\right)$$

$$= \frac{1}{Z_N}\sum_{\{n_{\vec{p}}\}_{\vec{p}\neq\vec{p}'}} e^{-\frac{1}{T}\sum_{\vec{p}\neq\vec{p}'}\varepsilon_{\vec{p}} n_{\vec{p}}} \sum_{l=1}^{N} e^{-\frac{1}{T}\varepsilon_{\vec{p}'} l}(-1)^{l+1} \sum_{n_{\vec{p}'=0}}^{min\,n[N-l,1]} e^{-\frac{1}{T}\varepsilon_{\vec{p}'} n_{\vec{p}'}}\,\delta\left(\sum_{\vec{p}\neq\vec{p}'} n_{\vec{p}} - (N - l - n_{\vec{p}'})\right)$$

$$= \frac{1}{Z_N}\sum_{l=1}^{N}(-1)^{l+1} e^{-\frac{1}{T}\varepsilon_{\vec{p}'} l} Z_{N-l}. \qquad (3.1.29)$$

Equation (3.1.16) can be written as [12]

$$Z_N = \sum_{l=1}^{N} y_l e^{-\frac{1}{T}\varepsilon_{\vec{p}'} l} Z_{N-l}, \qquad (3.1.30)$$

where $y_l = 1$ for the Bose-Einstein and $y_l = (-1)^{l+1}$ for Fermi-Dirac statistics.

3.2 GRAND CANONICAL ENSEMBLE

The grand thermodynamic potential is the function of variables (T, V, μ) [6, 7]. It describes the system which can exchanges particles as well as energy with heat reservoir. T is temperature of reservoir with which system is in contact, V is volume occupied by system and μ is chemical potential.

3.2.1 *General theory*

In statistical approach grand thermodynamic potential has form: $\Omega = \Omega(\{p_i\}, T, V, \mu)$- where set of $\{p_i\}$ is unknown. The thermodynamic potential of the grand canonical ensembe is obtained by the Legendre transform (2.4.1) from the fundamental thermodynamic potential $E = \sum_i p_i E_i$ using the entropy (3.1) and the number of particles $N = \sum_i p_i N_i$ [17]:

$$\Omega = E - TS - \mu N = \sum_i p_i \Omega_i, \qquad (3.2.1)$$

where $\Omega_i = E_i + T\ln p_i - \mu N_i$ in *i*-th micristate [6, 7, 12].
With the same algorithm as has been presented for canonical ensemble we obtain the set of probabilities [17]:

$$p_i = \frac{1}{Z} e^{-\frac{1}{T}(E_i - \mu N_i)}, \qquad (3.2.2)$$

$$Z = \sum_i e^{-\frac{1}{T}(E_i - \mu N_i)}, \qquad (3.2.2a)$$

where Z is partition function.

Using equation (3.2.2), we obtain:

$$S = \ln Z + \frac{1}{T}(E - \mu N), \qquad (3.2.3)$$

$$E = \frac{1}{Z}\sum_i E_i e^{-\frac{1}{T}(E_i - \mu N_i)}, \qquad (3.2.4)$$

$$N = \sum_i N_i\, e^{-\frac{1}{T}(E_i - \mu N_i)}, \qquad (3.2.5)$$

$$\Omega = -T\ln Z. \qquad (3.2.6)$$

3.2.2 *Ideal gas*

For the ideal gas energy and number of particles of the microstate are represented by the equations (3.1.9) and (3.1.10) respectively. It is important to emphasize that N is not exactly conserved in grand canonical ensemble, but mean N is constant for fixed variables of macroscopic state.

The statistical extensive averages can be written as [6, 7]

$$\langle A\rangle = \frac{1}{Z}\sum_{\{n_{\vec p}\}} A(\{n_{\vec p}\})e^{-\frac{1}{T}\sum_{\vec p} n_{\vec p}(\varepsilon_{\vec p}-\mu)}, \qquad (3.2.7)$$

$$Z = \sum_{\{n_{\vec p}\}} G(\{n_{\vec p}\})e^{-\frac{1}{T}\sum_{\vec p} n_{\vec p}(\varepsilon_{\vec p}-\mu)}. \qquad (3.2.8)$$

Let us derive partition function for ideal gas in grand-canonical ensemble for all types of statistics: Maxwell-Boltzmann, Bose-Einstein and Fermi-Dirac.

Classic statistics. Partition function for particles following Maxwell-Boltzmann statistics:

$$Z = \sum_{\{n_{\vec p}\}} \frac{1}{\prod_{\vec p} n_{\vec p}!} e^{-\frac{1}{T}\sum_{\vec p} n_{\vec p}(\varepsilon_{\vec p}-\mu)} = \sum_{n_1} \frac{1}{n_1!} e^{-\frac{1}{T}n_1(\varepsilon_1-\mu)} \sum_{n_2} \frac{1}{n_2!} e^{-\frac{1}{T}n_2(\varepsilon_2-\mu)} \sum_{n_3} \frac{1}{n_3!} e^{-\frac{1}{T}n_3(\varepsilon_3-\mu)} \dots . \qquad (3.2.9)$$

Every occupation number corresponds to number of particles in particular single-particle state and every particular set of $\{n_{\vec p}\}$ defines microstate. Using equation $\sum_n \frac{x^n}{n!} = e^x$, we obtain [6, 7]

$$Z = \prod_{\vec p} e^{e^{-\frac{1}{T}(\varepsilon_{\vec p}-\mu)}} = e^{\sum_{\vec p} e^{-\frac{1}{T}(\varepsilon_{\vec p}-\mu)}}. \qquad (3.2.9a)$$

Using (3.2.7), we obtain average of occupation numbers [6, 7]:

$$\langle n_{\vec p}\rangle = \frac{1}{Z}\sum_{\{n_{\vec p}\}} \frac{1}{\prod_{\vec p} n_{\vec p}!} n_{\vec p} e^{-\frac{1}{T}\sum_{\vec p} n_{\vec p}(\varepsilon_{\vec p}-\mu)}$$

$$= \frac{\sum_{n_1}\frac{1}{n_1!}e^{-\frac{1}{T}n_1(\varepsilon_1-\mu)}\sum_{n_2}\frac{1}{n_2!}e^{-\frac{1}{T}n_2(\varepsilon_2-\mu)}\sum_{n_3}\frac{1}{n_3!}e^{-\frac{1}{T}n_3(\varepsilon_3-\mu)}\dots\sum_{n_\alpha} n_\alpha\frac{1}{n_\alpha!}e^{-\frac{1}{T}n_\alpha(\varepsilon_\alpha-\mu)}\dots}{Z}$$

$$= \frac{\sum_{n_{\vec p}}\frac{1}{n_{\vec p}!}n_{\vec p}e^{-\frac{1}{T}n_{\vec p}(\varepsilon_{\vec p}-\mu)}}{\sum_{n_{\vec p}}\frac{1}{n_{\vec p}!}e^{-\frac{1}{T}n_{\vec p}(\varepsilon_{\vec p}-\mu)}} = -T\frac{\partial}{\partial\varepsilon_{\vec p}}\left(\ln\sum_{n_{\vec p}}\frac{1}{n_{\vec p}!}e^{-\frac{1}{T}n_{\vec p}(\varepsilon_{\vec p}-\mu)}\right) = -T\frac{\partial}{\partial\varepsilon_{\vec p}}\left(\sum_{n_{\vec p}} e^{-\frac{1}{T}(\varepsilon_{\vec p}-\mu)}\right) = e^{-\frac{1}{T}(\varepsilon_{\vec p}-\mu)}. \qquad (3.2.10)$$

In the grand canonical ensemble, the total number of particles N fluctuates near average value which can be obtained from equation (3.2.7):

$$\langle N\rangle = \frac{1}{Z}\sum_{\{n_{\vec p}\}} \frac{1}{\prod_{\vec p} n_{\vec p}!} N_{\{n_{\vec p}\}} e^{-\frac{1}{T}\sum_{\vec p} n_{\vec p}(\varepsilon_{\vec p}-\mu)}, \qquad (3.2.11)$$

where $N_{\{n_\alpha\}} = \sum_{\vec p} n_{\vec p}$ is sum over all single-particle states for particular microstate.

$$\langle N\rangle = \frac{1}{Z}\sum_{\{n_{\vec p}\}} \frac{1}{\prod_{\vec p} n_{\vec p}!}\sum_{\vec p} n_{\vec p}\, e^{-\frac{1}{T}\sum_{\vec p} n_{\vec p}(\varepsilon_{\vec p}-\mu)} = \sum_{\vec p} \frac{\sum_{n_{\vec p}}\frac{1}{n_{\vec p}!}n_{\vec p}e^{-\frac{1}{T}n_{\vec p}(\varepsilon_{\vec p}-\mu)}}{\sum_{n_{\vec p}}\frac{1}{n_{\vec p}!}e^{-\frac{1}{T}n_{\vec p}(\varepsilon_{\vec p}-\mu)}} = \sum_{\vec p}\langle n_{\vec p}\rangle. \qquad (3.2.12)$$

For average value of energy *E* of microstate we have:

$$\langle E\rangle = \frac{1}{Z}\sum_{\{n_{\vec p}\}} \frac{1}{\prod_{\vec p} n_{\vec p}!}\left[\sum_{\vec p} \varepsilon_{\vec p} n_{\vec p}\right] e^{-\frac{1}{T}\sum_{\vec p} n_{\vec p}(\varepsilon_{\vec p}-\mu)} = \sum_{\vec p} \varepsilon_{\vec p} \frac{\sum_{n_{\vec p}}\frac{1}{n_{\vec p}!}n_{\vec p}e^{-\frac{1}{T}n_{\vec p}(\varepsilon_{\vec p}-\mu)}}{\sum_{n_{\vec p}}\frac{1}{n_{\vec p}!}e^{-\frac{1}{T}n_{\vec p}(\varepsilon_{\vec p}-\mu)}} = \sum_{\vec p} \varepsilon_{\vec p}\langle n_{\vec p}\rangle. \qquad (3.2.13)$$

Let us obtain averages of intensive variables. We are not able to use (3.2.7) for such a variables. Thermodynamic potential defines system comprehensively, hence we can exert intensive averages from it. Substituting (3.2.9a) in (3.2.6), we obtain grand thermodynamic potential [6, 7]:

$$\Omega = -T\ln e^{\sum_{\vec{p}} e^{-\frac{1}{T}(\varepsilon_{\vec{p}}-\mu)}} = -T\sum_{\vec{p}} e^{-\frac{1}{T}(\varepsilon_{\vec{p}}-\mu)}. \quad (3.2.14)$$

Using (3.2.14) and (2.4.5), we obtain the pressure p [18]:

$$p = -\frac{\partial\Omega}{\partial V} = T\frac{\partial(\sum_{\vec{p}} e^{-\frac{1}{T}(\varepsilon_{\vec{p}}-\mu)})}{\partial V}$$

$$= T\sum_{\vec{p}} \frac{\partial e^{-\frac{1}{T}(\varepsilon_{\vec{p}}-\mu)}}{\partial V} = -\sum_{\vec{p}} e^{-\frac{1}{T}(\varepsilon_{\vec{p}}-\mu)}\frac{\partial\varepsilon_{\vec{p}}}{\partial V} = \frac{1}{3V}\sum_{\vec{p}} e^{-\frac{1}{T}(\varepsilon_{\vec{p}}-\mu)}\frac{p_{\vec{p}}{}^2}{\varepsilon_{\vec{p}}} = \frac{1}{3V}\sum_{\vec{p}}\frac{p_{\vec{p}}{}^2}{\varepsilon_{\vec{p}}}\langle n_{\vec{p}}\rangle . \quad (3.2.15)$$

Substituting (3.1.14) into equation (2.4.4), we obtain entropy:

$$S = -\frac{\partial\Omega}{\partial T} = \frac{\partial}{\partial T}(T\sum_{\vec{p}} e^{-\frac{1}{T}(\varepsilon_{\vec{p}}-\mu)}) = \sum_{\vec{p}} e^{-\frac{1}{T}(\varepsilon_{\vec{p}}-\mu)} + T(\sum_{\vec{p}} e^{-\frac{1}{T}(\varepsilon_{\vec{p}}-\mu)}\frac{(\varepsilon_{\vec{p}}-\mu)}{T^2})$$

$$= \ln Z + \frac{\sum_{\vec{p}}(\varepsilon_{\vec{p}}-\mu)e^{-\frac{1}{T}(\varepsilon_{\vec{p}}-\mu)}}{T}$$

$$= \ln Z + \frac{1}{T}(\langle E\rangle - \mu\langle N\rangle). \quad (3.2.16)$$

We have obtained equation equal to (3.2.3). It means that statistical thermodynamic potential Ω defined as $\Omega = -T\ln Z$ does correspond to thermodynamic potential Ω introduced in chapter 2.

In chapter III we obtained partition functions for grand-canonical and canonical ensembles for all statistics with discrete set of single-particle states. Every state is defined by energy $\varepsilon_{\vec{p}} = \sqrt{\vec{p}^2+m^2}$. We consider system in finite volume V, therefore momentum of quantum particles may obtain only discrete values: $p_i = \frac{2\pi n}{V^{1/3}}$ [19] , n=0,1,2.... , hence we obtain discrete energy states. The more volume become, the more dense states occur.

Let us focus on grand-canonical ensemble of classical particles in thermodynamic limit for large value of volume V. Thermodynamic potential is defined by the equation (3.2.14):

$$\Omega = -T\sum_{p} e^{-\frac{1}{T}(\varepsilon_{\vec{p}}-\mu)} = -T\frac{gV}{(2\pi)^3}\int d^3p e^{-\frac{1}{T}(\varepsilon_{\vec{p}}-\mu)}, \quad (3.2.17)$$

where $\varepsilon_p = \sqrt{\vec{p}^2+m^2}$. We calculate this integral over momentum in spherical coordinates: $d^3p = p^2\sin\Theta dp d\Theta d\varphi$. Integrating equation, we obtain

$$\Omega = -T\frac{4\pi gV}{(2\pi)^3}\int e^{-\frac{1}{T}(\varepsilon_p-\mu)}p^2dp = -T^2\frac{4\pi gV}{(2\pi)^3}e^{\frac{1}{T}\mu}m^2K_2(\frac{m}{T}), \quad (3.2.18)$$

where K_2 is modified Bessel function [20] of order 2 (see Appendix F).

The mean number of particles N can be written as

$$N = \sum_p\langle n_p\rangle = \frac{4\pi gV}{(2\pi)^3}\int e^{-\frac{1}{T}(\varepsilon_p-\mu)}p^2dp = -\frac{\partial}{\partial\mu}\left(-T\frac{4\pi gV}{(2\pi)^3}\int e^{-\frac{1}{T}(\varepsilon_p-\mu)}p^2dp\right) = -\frac{\partial\Omega}{\partial\mu}$$

$$= T\frac{4\pi gV}{(2\pi)^3}e^{\frac{1}{T}\mu}m^2K_2\left(\frac{m}{T}\right) = -\frac{\Omega}{T}. \quad (3.2.19)$$

For mean energy of the system we obtain equation with $\beta = \frac{1}{T}$:

$$E = \sum_p \varepsilon_p \langle n_p \rangle = \frac{4\pi gV}{(2\pi)^3} \int \varepsilon_p e^{-\beta(\varepsilon_p - \mu)} p^2 dp = e^{\beta\mu} \frac{\partial}{\partial \beta} (\frac{4\pi gV}{(2\pi)^3} \int e^{-\beta \varepsilon_p} p^2 dp)$$

$$= -e^{\beta\mu} \frac{\partial}{\partial \beta} \left(\frac{4\pi gV}{(2\pi)^3} \frac{m^2}{\beta} K_2(m\beta) \right) = e^{\beta\mu} \frac{4\pi gV}{(2\pi)^3} \frac{m^2}{\beta^2} K_2(m\beta) - e^{\beta\mu} \frac{4\pi gV}{(2\pi)^3} \frac{m^3}{\beta} \frac{\partial}{\partial m\beta} K_2(m\beta). \quad (3.2.20)$$

For modified Bessel function K_2 we have recurrence relations [20]:

$$\frac{\partial}{\partial x} K_2 = -K_1 - \frac{2}{x} K_2 \ . \quad (3.2.21)$$

Using (3.2.21), we obtain:

$$E = e^{\beta\mu} \frac{4\pi gV}{(2\pi)^3} \frac{m^2}{\beta^2} K_2(m\beta) + e^{\beta\mu} \frac{4\pi gV}{(2\pi)^3} \frac{m^3}{\beta} \left(K_1 + \frac{2}{m\beta} K_2 \right) = e^{\beta\mu} \frac{4\pi gV}{(2\pi)^3} \frac{m^3}{\beta} \left(K_1 + \frac{3}{m\beta} K_2 \right). \quad (3.2.22)$$

For obtaining entropy we use (2.4.4):

$$S = -\frac{\partial}{\partial T} \left(-T^2 \frac{4\pi gV}{(2\pi)^3} e^{\frac{1}{T}\mu} m^2 K_2 \left(\frac{m}{T} \right) \right) = e^{\beta\mu} \frac{4\pi gV}{(2\pi)^3} m^2 (K_1 m + 4 \frac{1}{\beta} K_2 - \mu K_2). \quad (3.2.23)$$

Finally, let us obtain pressure p. Substituting (3.2.18) into (2.4.5), we obtain

$$p = -\frac{\partial}{\partial V} (-T^2 \frac{4\pi gV}{(2\pi)^3} e^{\frac{1}{T}\mu} m^2 K_2 (\frac{m}{T})) = T^2 \frac{4\pi g}{(2\pi)^3} e^{\frac{1}{T}\mu} m^2 K_2 \left(\frac{m}{T} \right) = \frac{TN}{V} = -\frac{\Omega}{V}. \quad (3.2.24)$$

Quantum statistics. Let's us consider Bose-Einstein statistics. Partition function has form [6, 7]:

$$Z = \sum_{\{n_{\vec{p}}\}} e^{-\frac{1}{T} \sum_{\vec{p}} n_{\vec{p}} (\varepsilon_{\vec{p}} - \mu)} = \sum_{n_1} e^{-\frac{1}{T} n_1 (\varepsilon_1 - \mu)} \sum_{n_2} e^{-\frac{1}{T} n_2 (\varepsilon_2 - \mu)} \sum_{n_3} e^{-\frac{1}{T} n_3 (\varepsilon_3 - \mu)} \dots . \quad (3.2.25)$$

Every sum can be represented as $\sum_{n_{\vec{p}}} e^{-\frac{1}{T} n_{\vec{p}} (\varepsilon_{\vec{p}} - \mu)} = \frac{1}{1 - e^{-\frac{1}{T}(\varepsilon_{\vec{p}} - \mu)}}$,

then we obtain [24]

$$Z = \prod_{\vec{p}} \frac{1}{\left[1 - e^{-\frac{1}{T}(\varepsilon_{\vec{p}} - \mu)} \right]}. \quad (3.2.26)$$

For the mean occupation numbers we obtain equation [6]:

$$\langle n_{\vec{p}} \rangle = \frac{1}{Z} \sum_{\{n_{\vec{p}}\}} n_{\vec{p}} e^{-\frac{1}{T} \sum_{\vec{p}} n_{\vec{p}} (\varepsilon_{\vec{p}} - \mu)} = \frac{\sum_{n_1} e^{-\frac{1}{T} n_1 (\varepsilon_1 - \mu)} \sum_{n_2} e^{-\frac{1}{T} n_2 (\varepsilon_2 - \mu)} \sum_{n_3} e^{-\frac{1}{T} n_3 (\varepsilon_3 - \mu)} \dots \sum_{n_{\vec{p}}} n_{\vec{p}} e^{-\frac{1}{T} n_{\vec{p}} (\varepsilon_{\vec{p}} - \mu)} \dots}{Z}$$

$$= \frac{\sum_{n_{\vec{p}}} n_{\vec{p}} e^{-\frac{1}{T} n_{\vec{p}} (\varepsilon_{\vec{p}} - \mu)}}{\sum_{n_{\vec{p}}} e^{-\frac{1}{T} n_{\vec{p}} (\varepsilon_{\vec{p}} - \mu)}} = (-T) \frac{\partial}{\partial \varepsilon_{\vec{p}}} (\ln \sum_{n_{\vec{p}}} e^{-\frac{1}{T} n_{\vec{p}} (\varepsilon_{\vec{p}} - \mu)}) = (-T) \frac{\partial}{\partial \varepsilon_{\vec{p}}} (\ln \frac{1}{1 - e^{-\frac{1}{T}(\varepsilon_{\vec{p}} - \mu)}}) = \frac{1}{e^{\frac{1}{T}(\varepsilon_{\vec{p}} - \mu)} - 1} \, . \quad (3.2.27)$$

Thermodynamic potential has form [6]:

$$\Omega = -T \ln \prod_{\vec{p}} \frac{1}{\left[1-e^{-\frac{1}{T}(\varepsilon_{\vec{p}}-\mu)}\right]} = T \sum_{\vec{p}} \ln\left(1 - e^{-\frac{1}{T}(\varepsilon_{\vec{p}}-\mu)}\right). \qquad (3.2.28)$$

For Fermi-Dirac statistics occupation numbers assume values 0 and 1, hence we obtain [6]

$$Z = \sum_{\{n_{\vec{p}}\}} e^{-\frac{1}{T}\sum_{\vec{p}} n_{\vec{p}}(\varepsilon_{\vec{p}}-\mu)} = \prod_{\vec{p}}\left(1 + e^{-\frac{1}{T}(\varepsilon_{\vec{p}}-\mu)}\right). \qquad (3.2.29)$$

For mean occupation numbers we write [6]:

$$\langle n_{\vec{p}} \rangle = \frac{1}{Z}\sum_{\{n_{\vec{p}}\}} n_{\vec{p}} e^{-\frac{1}{T}\sum_{\vec{p}} n_{\vec{p}}(\varepsilon_{\vec{p}}-\mu)} = \frac{\sum_{n_{\vec{p}}} n_{\vec{p}} e^{-\frac{1}{T} n_{\vec{p}}(\varepsilon_{\vec{p}}-\mu)}}{\sum_{n_{\vec{p}}} e^{-\frac{1}{T} n_{\vec{p}}(\varepsilon_{\vec{p}}-\mu)}} = \frac{e^{-\frac{1}{T}(\varepsilon_{\alpha}-\mu)}}{1+e^{-\frac{1}{T}(\varepsilon_{\vec{p}}-\mu)}} = \frac{1}{e^{\frac{1}{T}(\varepsilon_{\vec{p}}-\mu)}+1}. \qquad (3.2.30)$$

CHAPTER IV. TRANSVERSE MOMENTUM DISTRIBUTION

4.1 DISTRIBUTION FUNCTION

In relativistic mechanics addition law of velocities moving along, for example along z-axis, is not linear. Therefore, we consider rapidity- quantity with linear addition law [3]:

$$y = \tanh^{-1}\beta = \frac{1}{2}\ln\frac{1+\beta}{1-\beta}, \qquad (4.1.1)$$

where $\beta = \frac{p_z}{E}$, therefore we can rewrite (4.1.1) as

$$y = \frac{1}{2}\ln\frac{E+p_z}{E-p_z}. \qquad (4.1.1a)$$

where E=$\sqrt{\vec{p}^2 + m^2}$. Let us consider the system moving with velocity β' along z-axis. Then three-dimensional momentum can be presented as addition of transverse and longitudinal components with respect to β'. Transverse momentum $p_T = \sqrt{p_x + p_y}$ do not differ with changing frame of reference, while longitudinal momentum p_z change according to transformation law [21]:

$$p_z = \frac{p'_z + \beta' E'}{\sqrt{1-\beta'^2}}. \qquad (4.1.2)$$

In order to obtain four-momentum in terms of p_T and y (which obeys linear addition law), we need to find p_z and E as functions $p_z(y, p_T)$ and $E(y, p_T)$. Let us exert $E(y, p_T)$ from equation (4.1.1a). For simplifying calculations we introduce
$m_T = \sqrt{m^2 + p_T{}^2}$, which is the transverse mass. Substituting m_T in (4.1.1a), we obtain

$$y = \frac{1}{2}\ln\frac{E+p_z}{E-p_z} = \ln\sqrt{\frac{E+p_z}{E-p_z}} = \ln\frac{E+p_z}{\sqrt{(E-p_z)(E+p_z)}} = \ln\frac{E+p_z}{\sqrt{(E^2-p_z{}^2)}} = \ln\frac{E+p_z}{m_T}, \qquad (4.1.3)$$

Then we obtain

$$\cosh y = \frac{1}{2}(e^y + e^{-y}) = \frac{1}{2}\left(\frac{E+p_z}{m_T} + \frac{m_T}{E+p_z}\right) = \frac{1}{2}\frac{E^2+2Ep_z+p_z{}^2+m_T{}^2}{m_T(E+p_z)} = \frac{2Ep_z+2E^2}{2m_T(E+p_z)} = \frac{E}{m_T}. \qquad (4.1.4)$$

From (4.1.14) we obtain $E(y, p_T)$:

$$E = m_T \cosh y, \qquad (4.1.5)$$

For p_z we have relation:

$$p_z = \sqrt{E^2 - m_T{}^2}. \qquad (4.1.6)$$

Substituting (4.1.5) into equation (4.1.6), we obtain

$$p_z = m_T \sinh y. \qquad (4.1.7)$$

Now we can write four-momentum in new set of variables [3]:

$$p^\mu = (m_T \cosh y, p_T \cos\varphi, p_T \sin\varphi, m_T \sinh y). \qquad (4.1.8)$$

Full number of particles can be defined by equation [10]:

$$N = \sum_{\vec{p},\sigma} \langle n_{\vec{p}\sigma} \rangle = \sum_{\sigma} \frac{V}{(2\pi)^3} \int d^3p \langle n_{\vec{p}\sigma} \rangle = \frac{gV}{(2\pi)^3} \int d^3p \langle n_{\vec{p}} \rangle = \frac{gV}{(2\pi)^3} \int dp_x dp_y dp_z \langle n_{\vec{p}} \rangle \,, \quad (4.1.9)$$

where $g = 2s + 1$ is degeneracy multiplicity of the state for particles with spin *s*. To proceed with (4.1.9) we use Lorentz-invariant $\frac{d^3p}{E}$ [3] (see Appendix D):

$$N = \frac{gV}{(2\pi)^3} \int E \frac{dp_x dp_y dp_z}{E} \langle n_{\vec{p}} \rangle, \quad (4.1.10)$$

On the other hand, full number of particles can be written with the use of derivative $\frac{d^3N}{d^3p}$:

$$N = \int E \frac{d^3N}{dp_x dp_y dp_z} \frac{dp_x dp_y dp_z}{E} \,. \quad (4.1.11)$$

Comparing (4.1.10) and (4.1.11), we obtain distribution function [10]:

$$E \frac{d^3N}{d^3p} = \frac{gV}{(2\pi)^3} E \langle n_{\vec{p}} \rangle, \quad (4.1.12)$$

where $d^3p = Ep_T dp_T dy d\varphi$ (see Appendix D). Finally, we obtain distribution function [22, 23]:

$$\frac{d^3N}{dp_T dy d\varphi} = \frac{gV}{(2\pi)^3} p_T \, m_T \cosh y \, \langle n_{\vec{p}} \rangle. \quad (4.1.13)$$

Let us consider ideal gas following Maxwell-Boltzmann statistics in grand-canonical ensemble. Subscribing equation (3.2.10) in (4.1.13), we obtain:

$$\frac{d^3N}{dp_T dy} = \frac{gV}{(2\pi)^2} p_T \, m_T \cosh y \, e^{-\frac{1}{T}(\varepsilon_{\vec{p}} - \mu)}. \quad (4.1.14)$$

Equation (4.1.14) is integrated over azimuthal angle φ.

CHAPTER V. QUANTUM STATISTICAL HADRON MODEL WITH EXECTLY CONSERVED STRANGENESS

In this chapter we consider the system of hadrons of different spices with exactly conserved number of strangeness and average conservation of barion and electric charges of the system.

5.1 SYSTEM WITH DIFFERENT SORTS OF PARTICLES IN THE GRAND CANONICAL ENSEMBLE

In chapter III we have obtained partition function for one sort of particles. Let us consider grand canonical ensemble for the set of species. Every species α is characterized by chemical potential μ_α. Here we consider ideal gas following Maxwell-Boltzmann statistics [4, 7]:

$$Z = \sum_{\{n_{\alpha p}\}} \frac{1}{\prod_{\alpha p} n_{\alpha p}!} e^{-\frac{1}{T}\sum_{\alpha p} n_{\alpha p}(\varepsilon_{\alpha p}-\mu_\alpha)} = \prod_\alpha \prod_p \frac{1}{n_{\alpha p}!} e^{-\frac{1}{T} n_{\alpha p}(\varepsilon_{\alpha p}-\mu_\alpha)} \,, \qquad (5.1.1)$$

where $\mu_\alpha = b_\alpha \mu_B + q_\alpha \mu_Q + s_\alpha \mu_S$. The mean occupation numbers and mean particle number of species α can be obtained as

$$\langle n_{\alpha' p'} \rangle = \frac{1}{Z} \sum_{\{n_{\alpha p}\}} \frac{1}{\prod_{\alpha p} n_{\alpha p}!} n_{\alpha' p'} e^{-\frac{1}{T}\sum_{\alpha p}(\varepsilon_{\alpha p}-\mu_\alpha) n_{\alpha p}} = e^{-\frac{1}{T}\left(\varepsilon_{\alpha' p'}-\mu_{\alpha'}\right)}, \qquad (5.1.2)$$

$$\langle n_{\alpha'} \rangle = \frac{1}{Z} \sum_{\{n_{\alpha p}\}} \frac{1}{\prod_{\alpha p} n_{\alpha p}!} \left[\sum_{p'} n_{\alpha' p'}\right] e^{-\frac{1}{T}\sum_{\alpha p}(\varepsilon_{\alpha p}-\mu_\alpha) n_{\alpha p}} = \sum_{p'} e^{-\frac{1}{T}\left(\varepsilon_{\alpha' p'}-\mu_{\alpha'}\right)} = \sum_{p'} \langle n_{\alpha' p'} \rangle. \qquad (5.1.3)$$

5.2 SYSTEM WITH DIFFERENT SORTS OF PARTICLES IN THE CANONICAL ENSEMBLE

5.2.1 Classical statistics: direct method of solving partition function.

The partition function of the system with different sorts of particles with exactly conserved strangeness by the canonical ensemble and with average conservation of baryon and electric charges by grand canonical ensemble can be written as [4]

$$Z_S = \sum_{\{n_{\alpha p}\}} \frac{1}{\prod_{\alpha p} n_{\alpha p}!} \delta\left(\sum_{\alpha p} s_\alpha n_{\alpha p} - S\right) e^{-\frac{1}{T}\sum_{\alpha p} n_{\alpha p}\varepsilon_{\alpha p}} \,, \qquad (5.2.1)$$

where $\varepsilon_{\alpha p} = \sqrt{\vec{p}^2 + m^2} + \mu_\alpha, \mu_\alpha = b_\alpha \mu_B + q_\alpha \mu_Q$. Statistical average of the extensive quantities can be defined as

$$\langle Q \rangle = \frac{1}{Z_S} \sum_{\{n_{\alpha p}\}} \frac{1}{\prod_{\alpha p} n_{\alpha p}!} Q_{\{n_{\alpha p}\}} \delta\left(\sum_{\alpha p} s_\alpha n_{\alpha p} - S\right) e^{-\frac{1}{T}\sum_{\alpha p} n_{\alpha p}\varepsilon_{\alpha p}} \,. \qquad (5.2.2)$$

Let us insert the identical unit $\sum_{\{n_\alpha\}} \prod_\alpha \delta\left(\sum_{\alpha p} n_{\alpha p} - n_\alpha\right) = 1$ into equation (5.2.1).

We obtain

$$Z_S = \sum_{\{n_{\alpha p}\}} \frac{1}{\prod_{\alpha p} n_{\alpha p}!} \delta(\sum_{\alpha p} s_\alpha n_{\alpha p} - S) e^{-\frac{1}{T}\sum_{\alpha p} n_{\alpha p}\varepsilon_{\alpha p}} \times \sum_{\{n_\alpha\}} \prod_\alpha \delta\left(\sum_{\alpha p} n_{\alpha p} - n_\alpha\right)$$

$$= \sum_{\{n_\alpha\}} \left[\prod_\alpha \sum_{\{n_{\alpha p}\}'} \frac{1}{\prod_p n_{\alpha p}!} e^{-\frac{1}{T}\sum_p n_{\alpha p}\varepsilon_{\alpha p}} \delta\left(\sum_{\alpha p} n_{\alpha p} - n_\alpha\right)\right] \delta(\sum_\alpha s_\alpha n_\alpha - S), \qquad (5.2.3)$$

where $\{n_{\alpha p}\}'$ is set of occupation numbers with fixed spices α, hence in brackets we obtain product over spices of functions $Z_{\alpha,n_\alpha} = \frac{1}{n_\alpha!}(\sum_p e^{-\frac{1}{T}\varepsilon_{\alpha p}})^{n_\alpha}$, which are not exact canonical partition functions (here we have $\varepsilon_{\alpha p} = \sqrt{\vec{p}^2 + m^2} + \mu_\alpha$, while for canonical partition function $\varepsilon_{\alpha p} = \sqrt{\vec{p}^2 + m^2}$). Substituting Z_{α,n_α} into equation (5.2.3), we obtain

$$Z_S = \sum_{\{n_\alpha\}} \left[\prod_\alpha \frac{1}{n_\alpha!}(e^{-\frac{1}{T}\varepsilon_{\alpha p}})^{n_\alpha}\right] \delta(\sum_\alpha s_\alpha n_\alpha - S) = \sum_{\{n_\alpha\}} \delta(\sum_\alpha s_\alpha n_\alpha - S) \prod_\alpha \frac{1}{n_\alpha!}(Z_\alpha^1)^{n_\alpha}, \qquad (5.2.4)$$

where

$$Z_\alpha^1 = \sum_p e^{-\frac{1}{T}\varepsilon_{\alpha p}} = \frac{g_\alpha V}{(2\pi)^3}\int d^3p e^{-\frac{1}{T}(\varepsilon_p - \mu_\alpha)} = T\frac{4\pi g_\alpha V}{(2\pi)^3} e^{\frac{1}{T}\mu_\alpha} m^2 K_2(\frac{m}{T}). \qquad (5.2.5)$$

We can proceed from the summation over spices α to summation over all possible values of strangeness $s \in [-s_{max}, s_{max}]$ for individual particle. Firstly, let us carry out summation over all $\{n_\alpha\}$. It can be accomplished by using integral representation of the Kronecker symbol $\delta(\sum_\alpha s_\alpha n_\alpha - S)$:

$$Z_S = \sum_{\{n_\alpha\}} \frac{1}{2\pi}\int d\varphi e^{-i(S - \sum_\alpha s_\alpha n_\alpha)\varphi} \prod_\alpha \frac{1}{n_\alpha!}(Z_\alpha^1)^{n_\alpha} \qquad (5.2.6)$$

Changing order of summation and integration, we obtain [24-26]

$$Z_S = \frac{1}{2\pi}\int d\varphi e^{-iS\varphi} \sum_{\{n_\alpha\}} \prod_\alpha \frac{1}{n_\alpha!}(Z_\alpha^1)^{n_\alpha} e^{is_\alpha n_\alpha \varphi}$$

$$= \frac{1}{2\pi}\int d\varphi e^{-iS\varphi} \prod_\alpha \sum_{n_\alpha} \frac{1}{n_\alpha!}(Z_\alpha^1)^{n_\alpha} e^{is_\alpha n_\alpha \varphi}$$

$$= \frac{1}{2\pi}\int d\varphi e^{-iS\varphi} \prod_\alpha e^{Z_\alpha^1 e^{is_\alpha \varphi}} = \frac{1}{2\pi}\int d\varphi e^{-iS\varphi} e^{\sum_\alpha Z_\alpha^1 e^{is_\alpha \varphi}}. \qquad (5.2.7)$$

Secondly, we introduce identity unit in form $\sum_s \delta_{s,s_\alpha} = 1$, where s_α stands for strangeness of the species α particles. Substituting this unit into equation (5.2.7), we have

$$Z_S = \frac{1}{2\pi}\int d\varphi e^{-iS\varphi} e^{\sum_\alpha \sum_s \delta_{s,s_\alpha} Z_\alpha^1 e^{is_\alpha \varphi}} = \frac{1}{2\pi}\int d\varphi e^{-iS\varphi} e^{\sum_s [\sum_\alpha \delta_{s,s_\alpha} Z_\alpha^1 e^{is_\alpha \varphi}]}$$

$$= \frac{1}{2\pi}\int d\varphi e^{-iS\varphi} e^{\sum_s S_s e^{is\varphi}}, \qquad (5.2.8)$$

where $S_s = \sum_\alpha \delta_{s,s_\alpha} Z_\alpha^1$.

After integrating over φ, we obtain partition function where summation is taken over all possible individual particle strangeness values [4]:

$$Z_S = \sum_{\{N_s\}} \delta\left(\sum_s sN_s - S\right) \prod_{s=-s_{max}}^{s_{max}} \frac{1}{N_s!} (S_s)^{N_s} \,. \tag{5.2.9}$$

For reducing number of summation from all values to only positive values, we introduce identity [4]

$$\prod_{s=1}^{s_{max}} \sum_{n_s} \delta(N_s - N_{-s} - n_s) = 1. \tag{5.2.10}$$

Substituting (5.2.10) into (5.2.9), we obtain

$$Z_S = \sum_{\{N_s\}} \delta\left(\sum_s sN_s - S\right) \prod_{s=-s_{max}}^{s_{max}} \frac{1}{N_s!} (S_s)^{N_s} \times \prod_{s=1}^{s_{max}} \sum_{n_s} \delta(N_s - N_{-s} - n_s)$$

$$= \sum_{\{N_s\}} \delta\left(\sum_{s=1}^{s_{max}} s(N_s - N_{-s}) - S\right) \prod_{s=1}^{s_{max}} \frac{S_s{}^{N_s} S_{-s}{}^{N_{-s}}}{N_s! N_{-s}!} \times \prod_{s=1}^{s_{max}} \sum_{n_s} \delta(N_s - N_{-s} - n_s) \mathrm{Z}_0$$

$$= Z_0 \prod_{s=1}^{s_{max}} \sum_{n_s} \delta\left(\sum_{s=1}^{s_{max}} sn_s - S\right) \prod_{s=1}^{s_{max}} \sum_{N_s} \sum_{N_{-s}} \frac{S_s{}^{N_s} S_{-s}{}^{N_{-s}}}{N_s! N_{-s}!} \delta(N_s - N_{-s} - n_s) \,, \tag{5.2.11}$$

where

$$Z_0 = \sum_{N_0} \frac{1}{N_0!} \left(\sum_\alpha \delta_{0,s_\alpha} Z_\alpha^1\right)^{N_0} = e^{\left(\sum_\alpha \delta_{0,s_\alpha} Z_\alpha^1\right)}. \tag{5.2.12}$$

Let us carry out summation over N_s and N_{-s}:

$$\sum_{N_s} \sum_{N_{-s}} \frac{S_s^{N_s} \; S_{-s}^{N_{-s}}}{N_s! N_{-s}!} \delta\,(N_s - N_{-s} - n_s) = \sum_{N_s} \frac{S_s^{N_s} \; S_{-s}^{(N_s - n_s)}}{N_s!(N_s - n_s)!} = S_{-s}^{-n_s} \sum_{N_s} \frac{(S_s\, S_{-s})^{N_s}}{N_s!(N_s - n_s)!}$$

$$= \sqrt{\frac{S_s^{n_s}}{S_{-s}^{n_s}}}\, I_{-n_s}\left(2\sqrt{S_s\, S_{-s}}\right), \tag{5.2.13}$$

where I_{-n_s} is modified Bessel function. Here we have used series expansion of modified Bessel function. Substituting (5.2.13) into equation (5.2.12), we obtain [4]

$$Z_S = Z_0 \prod_{s=1}^{s_{max}} \sum_{n_s} \delta\left(\sum_{s=1}^{s_{max}} sn_s - S\right) \prod_{s=1}^{s_{max}} \sqrt{\frac{S_s^{n_s}}{S_{-s}^{n_s}}}\, I_{-n_s}\left(2\sqrt{S_s\, S_{-s}}\right). \tag{5.2.14}$$

5.2.2 Classical statistics: method of recurrence relation

Let us write the initial partition function (5.2.1) as the product of the components: summation over species with individual positive, negative and zero strangeness charge s [4]:

$$Z_S = \left[\sum_{\{n_{\alpha p}\}_{\alpha \in M_0}} \frac{1}{\prod_{\alpha p} n_{\alpha p}!} e^{-\frac{1}{T}\sum_{\alpha p} n_{\alpha p} \varepsilon_{\alpha p}}\right] \times \left[\sum_{\{n_{\alpha p}\}_{\alpha \in M_+}} \frac{1}{\prod_{\alpha p} n_{\alpha p}!} e^{-\frac{1}{T}\sum_{\alpha p} n_{\alpha p} \varepsilon_{\alpha p}}\right]$$

$$\times \left[\sum_{\{n_{\alpha p}\}_{\alpha \in M_-}} \frac{1}{\prod_{\alpha p} n_{\alpha p}!} e^{-\frac{1}{T}\sum_{\alpha p} n_{\alpha p} \varepsilon_{\alpha p}}\right] \delta\left(\sum_{\alpha p} s_\alpha n_{\alpha p} - S\right), \tag{5.2.15}$$

where M_+, M_- and M_0 corresponds to subsets with positive, negative and zero individual strangeness s respectfully.

We introduce notation S_+ and S_- with following definitions:

$$S_\pm = \sum_{\alpha\in M_\pm}\sum_p |s_\alpha| n_{\alpha p} \ . \tag{5.2.16}$$

For $S_\pm$ we can write identity units [4]:

$$\sum_{S_\pm=0}^{\infty}\delta\left(\sum_{\alpha\in M_\pm}\sum_p |s_\alpha| n_{\alpha p} - S_\pm\right) = 1. \tag{5.2.17}$$

Substituting both units into (5.2.15), we obtain

$$Z_S = \left[\sum_{\{n_{\alpha p}\}_{\alpha\in M_0}} \frac{1}{\prod_{\alpha p} n_{\alpha p}!} e^{-\frac{1}{T}\sum_{\alpha p} n_{\alpha p}\varepsilon_{\alpha p}}\right]$$

$$\times\left[\sum_{\{n_{\alpha p}\}_{\alpha\in M_+}} \frac{1}{\prod_{\alpha p} n_{\alpha p}!} e^{-\frac{1}{T}\sum_{\alpha p} n_{\alpha p}\varepsilon_{\alpha p}} \sum_{S_+=0}^{\infty}\delta\left(\sum_{\alpha\in M_+}\sum_p |s_\alpha| n_{\alpha p} - S_+\right)\right]$$

$$\times\left[\sum_{\{n_{\alpha p}\}_{\alpha\in M_-}} \frac{1}{\prod_{\alpha p} n_{\alpha p}!} e^{-\frac{1}{T}\sum_{\alpha p} n_{\alpha p}\varepsilon_{\alpha p}} \sum_{S_-=0}^{\infty}\delta\left(\sum_{\alpha\in M_-}\sum_p |s_\alpha| n_{\alpha p} - S_-\right)\right]\delta(S_+ - S_- - S) \ . \tag{5.2.18}$$

Let us introduce new partition functions:

$$Z_{S_+} = \sum_{\{n_{\alpha p}\}_{\alpha\in M_+}} \frac{1}{\prod_{\alpha p} n_{\alpha p}!} e^{-\frac{1}{T}\sum_{\alpha p} n_{\alpha p}\varepsilon_{\alpha p}}\delta\left(\sum_{\alpha\in M_+}\sum_p |s_\alpha| n_{\alpha p} - S_+\right), \tag{5.2.19}$$

$$Z_{S_-} = \sum_{\{n_{\alpha p}\}_{\alpha\in M_-}} \frac{1}{\prod_{\alpha p} n_{\alpha p}!} e^{-\frac{1}{T}\sum_{\alpha p} n_{\alpha p}\varepsilon_{\alpha p}}\delta\left(\sum_{\alpha\in M_-}\sum_p |s_\alpha| n_{\alpha p} - S_-\right), \tag{5.2.20}$$

$$Z_0 = \sum_{\{n_{\alpha p}\}_{\alpha\in M_0}} \frac{1}{\prod_{\alpha p} n_{\alpha p}!} e^{-\frac{1}{T}\sum_{\alpha p} n_{\alpha p}\varepsilon_{\alpha p}}. \tag{5.2.21}$$

Using (5.2.16), we can rewrite equation (5.2.18) as

$$Z_S = Z_0 \sum_{S_+=0}^{\infty}\sum_{S_-=0}^{\infty} Z_{S_+} Z_{S_-}\delta(S_+ - S_- - S). \tag{5.2.22}$$

Let us find recurrence relations for $Z_{S_\pm}$:

$$Z_{S_\pm} = \sum_{\{n_{\alpha p}\}_{\alpha\in M_\pm}} \frac{1}{\prod_{\alpha p} n_{\alpha p}!} e^{-\frac{1}{T}\sum_{\alpha p} n_{\alpha p}\varepsilon_{\alpha p}}\delta\left(\sum_{\alpha\in M_\pm}\sum_p |s_\alpha| n_{\alpha p} - S_\pm\right)\times\frac{1}{S_\pm}\sum_{\alpha\in M_\pm}\sum_p |s_\alpha| n_{\alpha p}$$

$$= \frac{1}{S_\pm}\sum_{\alpha\in M_\pm}\sum_p \sum_{\{n_{\alpha p}\}_{\alpha\in M_\pm}} |s_\alpha| n_{\alpha p}\frac{1}{\prod_{\alpha p} n_{\alpha p}!} e^{-\frac{1}{T}\sum_{\alpha p} n_{\alpha p}\varepsilon_{\alpha p}}\delta\left(\sum_{\alpha\in M_\pm}\sum_p |s_\alpha| n_{\alpha p} - S_\pm\right)$$

$$= \frac{1}{S_\pm}\sum_{\alpha'\in M_\pm}\sum_{p'}\sum_{\{n_{\alpha p}\}_{\alpha\in M_\pm}^{\alpha\neq\alpha', p\neq p'}} \frac{1}{\prod_{\alpha p} n_{\alpha p}!} e^{-\frac{1}{T}\sum_{\alpha p} n_{\alpha p}\varepsilon_{\alpha p}}$$

$$\times\sum_{n_{\alpha' p'}} |s_{\alpha'}| n_{\alpha' p'} e^{-\frac{1}{T} n_{\alpha p}\varepsilon_{\alpha p}}\delta\left(\sum_{\alpha\in M_\pm}\sum_p |s_\alpha| n_{\alpha p} - S_\pm\right). \tag{5.2.23}$$

To proceed we use relation [4] (see Appendix G):

$$\sum_{n_{\alpha p}=0}^{\infty} n_{\alpha p} \frac{1}{n_{\alpha p}!} e^{-\frac{1}{T} n_{\alpha p}\varepsilon_{\alpha p}} \delta(\sum_{\alpha \in M_{\pm}} \sum_p |s_\alpha| n_{\alpha p} - S_{\pm})$$

$$= e^{-\frac{1}{T}\varepsilon_{\alpha p}} \sum_{n_{\alpha p}=0}^{\infty} \frac{1}{n_{\alpha p}!} e^{-\frac{1}{T}\varepsilon_{\alpha p} n_{\alpha p}} \delta(\sum_{\alpha \in M_{\pm}} \sum_p |s_\alpha| n_{\alpha p} - (S_{\pm} - |s_\alpha|). \quad (5.2.24)$$

Substituting (5.2.24) into equation (5.2.23), we obtain

$$Z_{S_{\pm}} = \frac{1}{S_{\pm}} \sum_{\alpha' \in M_{\pm}} |s_{\alpha'}| \sum_{p'} e^{-\frac{1}{T}\varepsilon_{\alpha' p'}} \sum_{\{n_{\alpha p}\}_{\alpha \in M_{\pm}}} \frac{1}{\prod_{\alpha p} n_{\alpha p}!} e^{-\frac{1}{T}\sum_{\alpha p} n_{\alpha p}\varepsilon_{\alpha p}} \delta(\sum_{\alpha \in M_{\pm}} \sum_p |s_\alpha| n_{\alpha p} - (S_{\pm} - |s_{\alpha'}|))$$

$$= \frac{1}{S_{\pm}} \sum_{\alpha' \in M_{\pm}} |s_{\alpha'}| \sum_{p'} e^{-\frac{1}{T}\varepsilon_{\alpha' p'}} Z_{S_{\pm}-|s_{\alpha'}|} = \frac{1}{S_{\pm}} \sum_{\alpha' \in M_{\pm}} |s_{\alpha'}| Z_{\alpha'}^{1} Z_{S_{\pm}-|s_{\alpha'}|} \,. \quad (5.2.25)$$

Inserting $\sum_s \delta_{s,\, s_\alpha} = 1$ into (5.2.25), we obtain

$$Z_{S_{\pm}} = \frac{1}{S_{\pm}} \sum_{\alpha' \in M_{\pm}} |s_{\alpha'}| Z_{\alpha'}^{1} Z_{S_{\pm}-|s_{\alpha'}|} \times \sum_s \delta_{s,\, s_{\alpha'}} = \frac{1}{S_{\pm}} \sum_s \sum_{\alpha' \in M_{\pm}} \delta_{s,\, s_{\alpha'}} |s_{\alpha'}| Z_{\alpha'}^{1} Z_{S_{\pm}-|s_{\alpha'}|}$$

$$= \frac{1}{S_{\pm}} \sum_s s S_{\pm s} Z_{S_{\pm}-s} \,, \quad (5.2.26)$$

where $S_{\pm s} = \sum_{\alpha \in M_{\pm}} \delta_{s,\, s_\alpha} Z_{\alpha}^{1}$.

The mean occupation number can be written as

$$\langle n_{\alpha' p'} \rangle_S = \frac{1}{Z_S} \sum_{\{n_{\alpha p}\}} \frac{1}{\prod_{\alpha p} n_{\alpha p}!} n_{\alpha' p'} e^{-\frac{1}{T}\sum_{\alpha p} \varepsilon_{\alpha p} n_{\alpha p}} \delta(\sum_{\alpha p} s_\alpha n_{\alpha p} - S)$$

$$= \frac{1}{Z_S} \sum_{\{n_{\alpha p}\}_{\alpha \neq \alpha', p \neq p'}} \frac{1}{\prod_{\alpha p} n_{\alpha p}!} e^{-\frac{1}{T}\sum_{\alpha p} \varepsilon_{\alpha p} n_{\alpha p}} \sum_{n_{\alpha' p'}} \frac{1}{n_{\alpha p}!} n_{\alpha' p'} e^{-\frac{1}{T}\varepsilon_{\alpha p} n_{\alpha p}} \delta(\sum_{\alpha p} s_\alpha n_{\alpha p} - S). \quad (5.2.27)$$

Applying relation (5.2.24), we obtain:

$$\langle n_{\alpha' p'} \rangle_S = \frac{1}{Z_S} e^{-\frac{1}{T}\varepsilon_{\alpha' p'}} \sum_{\{n_{\alpha p}\}} \frac{1}{\prod_{\alpha p} n_{\alpha p}!} e^{-\frac{1}{T}\sum_{\alpha p} \varepsilon_{\alpha p} n_{\alpha p}} \delta(\sum_{\alpha p} s_\alpha n_{\alpha p} - (S - s_{\alpha'}))$$

$$= \frac{1}{Z_S} e^{-\frac{1}{T}\varepsilon_{\alpha' p'}} Z_{S-s_{\alpha'}} \,. \quad (5.2.28)$$

It is easy to see that mean occupation numbers for particles with zero individual strangeness ($\alpha \in M_0$) can be obtained by the simplified equation:

$$\langle n_{\alpha' p'} \rangle_S = \frac{1}{Z_S} e^{-\frac{1}{T}\varepsilon_{\alpha' p'}} Z_{S-0} = e^{-\frac{1}{T}\varepsilon_{\alpha' p'}}. \quad (5.2.29)$$

The mean particle number for particular species α':

$$\langle n_{\alpha'}\rangle_S = \frac{1}{Z_S}\sum_{\{n_{\alpha p}\}}\frac{1}{\prod_{\alpha p} n_{\alpha p}!}\left[\sum_{p'} n_{\alpha' p'}\right]e^{-\frac{1}{T}\sum_{\alpha p}\varepsilon_{\alpha p}n_{\alpha p}}\delta\left(\sum_{\alpha p}s_\alpha n_{\alpha p}-S\right)$$

$$=\frac{1}{Z_S}\sum_{p'}\sum_{\{n_{\alpha p}\}_{\alpha\neq\alpha',p\neq p'}}\frac{1}{\prod_{\alpha p}n_{\alpha p}!}e^{-\frac{1}{T}\sum_{\alpha p}\varepsilon_{\alpha p}n_{\alpha p}}\sum_{n_{\alpha' p'}}\frac{1}{n_{\alpha' p'}!}n_{\alpha' p'}e^{-\frac{1}{T}\varepsilon_{\alpha' p'}n_{\alpha' p'}}\delta\left(\sum_{\alpha p}s_\alpha n_{\alpha p}-S\right)$$

$$=\frac{1}{Z_S}\left[\sum_{p'}e^{-\frac{1}{T}\varepsilon_{\alpha' p'}}\right]Z_{S-s_{\alpha'}}=\sum_{p'}\langle n_{\alpha' p'}\rangle_S. \tag{5.2.30}$$

For the mean number for particle with individual strangeness s, we can write equation:

$$\langle N_s\rangle_S=\frac{1}{Z_S}\sum_{\{n_{\alpha p}\}}\frac{1}{\prod_{\alpha p}n_{\alpha p}!}\left[\sum_\alpha\delta_{s,s_\alpha}\sum_{p'}n_{\alpha' p'}\right]e^{-\frac{1}{T}\sum_{\alpha p}\varepsilon_{\alpha p}n_{\alpha p}}\delta\left(\sum_{\alpha p}s_\alpha n_{\alpha p}-S\right)$$

$$=\frac{1}{Z_S}\left[\sum_\alpha\delta_{s,s_\alpha}\sum_{p'}e^{-\frac{1}{T}\varepsilon_{\alpha' p'}}Z_{S-s_\alpha}\right]=\sum_\alpha\delta_{s,s_\alpha}\sum_{p'}\langle n_{\alpha' p'}\rangle_S\ . \tag{5.2.31}$$

5.2.3 Quantum statistics: method of recurrence relation

For the particles that follows quantum statistics in canonical partition function with exact conservation of net strangeness can represented by the equation [4, 7]:

$$Z_S=\sum_{\{n_{\alpha p}\}}\delta\left(\sum_{\alpha p}s_\alpha n_{\alpha p}-S\right)e^{-\frac{1}{T}\sum_{\alpha p}n_{\alpha p}\varepsilon_{\alpha p}}\ . \tag{5.2.32}$$

It can be rewritten in terms of $Z_{S_\mp}$ and Z_0:

$$Z_S=Z_0\sum_{S_+=0}^{\infty}\sum_{S_-=0}^{\infty}Z_{S_+}Z_{S_-}\delta(S_+-S_--S), \tag{5.2.33}$$

where

$$Z_{S_\mp}=\sum_{\{n_{\alpha p}\}_{\alpha\in M_\mp}}e^{-\frac{1}{T}\sum_{\alpha p}n_{\alpha p}\varepsilon_{\alpha p}}\delta\left(\sum_{\alpha\in M_\mp}\sum_p|s_\alpha|n_{\alpha p}-S_\mp\right)\ , \tag{5.2.34}$$

$$Z_0=\sum_{\{n_{\alpha p}\}_{\alpha\in M_0}}e^{-\frac{1}{T}\sum_{\alpha p}n_{\alpha p}\varepsilon_{\alpha p}}\ . \tag{5.2.35}$$

Let us obtain recurrence equations for $Z_{S_\mp}$ [4]:

$$Z_{S_\mp}=\sum_{\{n_{\alpha p}\}_{\alpha\in M_\mp}}e^{-\frac{1}{T}\sum_{\alpha p}n_{\alpha p}\varepsilon_{\alpha p}}\delta\left(\sum_{\alpha\in M_\mp}\sum_p|s_\alpha|n_{\alpha p}-S_\mp\right)\times\frac{1}{S_\mp}\sum_{\alpha\in M_\mp}\sum_{\alpha'\in M_\mp}\sum_{p'}|s_{\alpha'}|n_{\alpha' p'}$$

$$=\frac{1}{S_\pm}\sum_{\alpha'\in M_\pm}\sum_{p'}\sum_{\{n_{\alpha p}\}_{\alpha\in M_\pm}^{\alpha\neq\alpha',p\neq p'}}e^{-\frac{1}{T}\sum_{\alpha p}n_{\alpha p}\varepsilon_{\alpha p}}\sum_{n_{\alpha' p'}}|s_{\alpha'}|n_{\alpha' p'}e^{-\frac{1}{T}n_{\alpha p}\varepsilon_{\alpha p}}\delta\left(\sum_{\alpha\in M_\pm}\sum_p|s_\alpha|n_{\alpha p}-S_\pm\right). \tag{5.2.36}$$

To proceed we introduce the equation [4] (see Appendix G):

$$\sum_{n_{\alpha p}}^{K_\alpha} n_{\alpha' p'} e^{-\frac{1}{T} n_{\alpha p}\varepsilon_{\alpha p}} \delta(\sum_{\alpha\in M_\pm}\sum_p |s_\alpha| n_{\alpha p} - S_\pm)$$

$$= \sum_{l=1}^{\left[\frac{S_\pm}{|s_\alpha|}\right]} y_{\alpha l} e^{-\frac{1}{T} l \varepsilon_{\alpha p}} \sum_{n_{\alpha p=0}}^{min\left\{\left[\frac{S_\pm}{|s_\alpha|}\right]-l,\, K_\alpha\right\}} e^{-\frac{1}{T} n_{\alpha p}\varepsilon_{\alpha p}} \delta(\sum_{\alpha\in M_\pm}\sum_p |s_\alpha| n_{\alpha p} - (S_\pm - l|s_{\alpha'}|) \quad (5.2.37)$$

$$= Z_{S_\mp} = \frac{1}{S_\pm}\sum_{\alpha'\in M_\pm}\sum_{p'}\sum_{\{n_{\alpha p}\}_{\alpha\in M_\mp}} e^{-\frac{1}{T}\sum_{\alpha p} n_{\alpha p}\varepsilon_{\alpha p}} \sum_{l=1}^{\left[\frac{S_\pm}{|s_\alpha|}\right]} y_{\alpha l} e^{-\frac{1}{T} l\varepsilon_{\alpha p}} \delta(\sum_{\alpha\in M_\pm}\sum_p |s_\alpha| n_{\alpha p} - (S_\pm - l|s_{\alpha'}|)$$

$$= \frac{1}{S_\pm}\sum_{l=1}^{\left[\frac{S_\pm}{|s_\alpha|}\right]} \sum_{\alpha'\in M_\pm} y_{\alpha l} Z_{S_\mp - l|s_{\alpha'}|}\left[\sum_{p'} e^{-\frac{1}{T} l\varepsilon_{\alpha p}}\right]. \quad (5.2.38)$$

Introducing $Z_\alpha^l = y_{\alpha l}\sum_{p'} e^{-\frac{1}{T} l\varepsilon_{\alpha p}}$, we can rewrite equation (5.2.38) as

$$Z_{S_\mp} = \frac{1}{S_\pm}\sum_{l=1}^{\left[\frac{S_\pm}{|s_\alpha|}\right]} \sum_{\alpha'\in M_\pm} Z_{S_\mp - l|s_{\alpha'}|} Z_\alpha^l. \quad (5.2.38a)$$

Partition function for zero strangeness Z_0 can be presented as

$$Z_0 = \sum_{\{n_{\alpha p}\}_{\alpha\in M_0}} e^{-\frac{1}{T}\sum_{\alpha p} n_{\alpha p}\varepsilon_{\alpha p}} = \prod_{\alpha\in M_0}\sum_{\{n_{\alpha p}\}'_{\alpha\in M_0}} e^{-\frac{1}{T}\sum_p n_{\alpha p}\varepsilon_{\alpha p}} \times \sum_{\{n_\alpha\}}\prod_\alpha \delta\left(\sum_{\alpha p} n_{\alpha p} - n_\alpha\right)$$

$$= \prod_{\alpha\in M_0}\sum_{n_\alpha}\sum_{\{n_{\alpha p}\}'_{\alpha\in M_0}} e^{-\frac{1}{T}\sum_p n_{\alpha p}\varepsilon_{\alpha p}} \delta\left(\sum_{\alpha p} n_{\alpha p} - n_\alpha\right). \quad (5.2.39)$$

Z_0 can be written in following form [4] (see Appendix H):

$$Z_0 = \prod_{\alpha\in M_0}\sum_{n_\alpha}\sum_{\{N_{\alpha l}\}'_{\alpha\in M_0}} \delta\left(\sum_{l=1}^{n_\alpha} lN_{\alpha l} - n_\alpha\right)\prod_{l=1}^{n_\alpha}\frac{(Z_\alpha^l)^{N_{\alpha l}}}{l^{N_{\alpha l}} N_{\alpha l}!}$$

$$= \prod_{\alpha\in M_0}\sum_{\{N_{\alpha l}\}'_{\alpha\in M_0}}\sum_{n_\alpha} \delta\left(\sum_{l=1}^{n_\alpha} lN_{\alpha l} - n_\alpha\right)\prod_{l=1}^{n_\alpha}\frac{(Z_\alpha^l)^{N_{\alpha l}}}{l^{N_{\alpha l}} N_{\alpha l}!} = \prod_{\alpha\in M_0}\sum_{\{N_{\alpha l}\}'_{\alpha\in M_0}}\prod_{l=1}^{\infty}\frac{(Z_\alpha^l)^{N_{\alpha l}}}{l^{N_{\alpha l}} N_{\alpha l}!}$$

$$= \prod_{\alpha\in M_0}\prod_{l=1}^{\infty}\sum_{N_{\alpha l}}\frac{(Z_\alpha^l)^{N_{\alpha l}}}{l^{N_{\alpha l}} N_{\alpha l}!} = \prod_{\alpha\in M_0} e^{\sum_{l=1}^{\infty}\frac{Z_\alpha^l}{l}}. \quad (5.2.40)$$

Let us derive some ensemble averages. Mean occupation numbers for species $\alpha \in M_\pm$ can be obtained with the use of equation (5.2.33):

$$\langle n_{\alpha p}\rangle_S = \frac{1}{Z_S} Z_0 \sum_{S_+=0}^{\infty}\sum_{S_-=0}^{\infty} Z_{S_+} Z_{S_-} n_{\alpha p}\delta(S_+ - S_- - S)$$

$$= \frac{1}{Z_S} Z^0 \sum_{S_\mp=0}^{\infty} Z_{S_\mp}\sum_{S_\pm=0}^{\infty}\left[\sum_{\{n_{\alpha p}\}_{\alpha\in M_\pm}} n_{\alpha' p'} e^{-\frac{1}{T}\sum_{\alpha p} n_{\alpha p}\varepsilon_{\alpha p}} \delta(\sum_{\alpha\in M_\pm}\sum_p |s_\alpha| n_{\alpha p} - S_\pm)\right]\delta(S_+ - S_- - S)$$

$$= \frac{1}{Z_S} Z_0 \sum_{S_\mp=0}^{\infty} Z_{S_\mp}\sum_{S_\pm}^{\infty}\sum_{l=1}^{\left[\frac{S_\pm}{|s_\alpha|}\right]} y_{\alpha l} e^{-\frac{1}{T} l\varepsilon_{\alpha p}} Z_{S_\pm - l|s_\alpha|}\delta(S_+ - S_- - S). \quad (5.2.41)$$

For species $\alpha \in M_0$ mean occupation numbers can be obtained as [4]

$$\langle n_{\alpha' p'} \rangle_S = \frac{1}{Z_S} \sum_{S_+=0}^{\infty} \sum_{S_-=0}^{\infty} Z_{S_+} Z_{S_-} \sum_{\{n_{\alpha p}\}_{\alpha \in M_0}} e^{-\frac{1}{T}\sum_{\alpha p} n_{\alpha p} \varepsilon_{\alpha p}} n_{\alpha' p'} \delta(S_+ - S_- - S)$$

$$= \frac{1}{Z_0} \sum_{\{n_{\alpha p}\}_{\alpha \in M_0}} n_{\alpha' p'} e^{-\frac{1}{T}\sum_{\alpha p} n_{\alpha p} \varepsilon_{\alpha p}} = -T \frac{\partial \ln Z_0}{\partial \varepsilon_{\alpha' p'}} . \quad (5.2.42)$$

Substituting (5.2.40) into equation (5.4.42), we obtain:

$$\langle n_{\alpha' p'} \rangle_S = -T \frac{\partial}{\partial \varepsilon_{\alpha' p'}} \left(\sum_{\alpha l} ln \left(e^{\frac{z_\alpha^l}{l}} \right) \right) = -\frac{T}{l} \left(\sum_l \frac{\partial z_\alpha^l}{\partial \varepsilon_{\alpha' p'}} \right) = \sum_{l=1}^{\infty} y_{\alpha l} e^{-\frac{1}{T} l \varepsilon_{\alpha' p'}}. \quad (5.2.43)$$

When $\left| e^{-\frac{1}{T} l \varepsilon_{\alpha' p'}} \right| < 1$, we obtain

$$\langle n_{\alpha' p'} \rangle_S = \frac{1}{1 - e^{-\frac{1}{T}\varepsilon_{\alpha' p'}}} - 1 = \frac{1}{e^{\frac{1}{T}\varepsilon_{\alpha' p'}} - 1} \text{ for the Bose-Einstein statistics,} \quad (5.2.44)$$

$$\langle n_{\alpha' p'} \rangle_S = -\frac{1}{1 + e^{-\frac{1}{T}\varepsilon_{\alpha' p'}}} + 1 = \frac{1}{e^{\frac{1}{T}\varepsilon_{\alpha' p'}} + 1} \text{ for the Fermi-Dirac statistics.} \quad (5.2.45)$$

As for classical statistics for quantum one the mean numbers of particles of sort α and the mean number for particles with individual strangeness s can be obtained by the equations:

$$\langle n_\alpha \rangle_S = \sum_{p'} \langle n_{\alpha p} \rangle_S, \quad (5.2.46)$$

$$\langle N_s \rangle_S = \sum_\alpha \delta_{s, s_\alpha} \sum_p \langle n_{\alpha p} \rangle_S. \quad (5.2.47)$$

CONCLUSION

We have considered the general formulation of the equilibrium statistical mechanics in canonical and grand canonical ensembles. The general equations for the partition function and ensemble averages of the canonical and grand canonical have been obtained.

We have calculated the partition functions and ensemble averages for ideal gas of particles following Maxwell-Boltzmann, Bose-Einstein and Fermi-Dirac statistics in the canonical and grand canonical ensembles. In the calculations the representation of occupation numbers has been applied. The partition function and ensemble averages in the case of classical particles in the grand canonical ensemble have been integrated over momentum. Eventual equations are represented with the use of Modified Bessel functions as the functions of variables of state.

We have considered the quantum statistical hadron model with exactly conserved strange charge of the system. We have obtained the exact solution of the partition function using the direct and the recurrence equation methods. Ensembles averages are obtained for particles carrying strange charge and for particles with neutral strangeness. Obtained equations can be applied for evaluation of the particle yield in heavy ion collisions and further description of the observed strangeness production rise.

APPENDIX A

For proper understanding the relation between energy (E^c,E^{lab}) and $\sqrt{s}$, $\sqrt{s_{NN}}$, let us consider an example. For two colliding protons $\sqrt{s}$ = 5 Tev. It meant that E^c=2.5 Tev (from the equation (1.2.3a)). To what energy the ion of Au will be accelerated can be calculated by multiplying E^c by amount of protons in Au. Therefore, the $E^c_{Au} = 2.5 \times 79 = 197.5$ Tev. Then, $\sqrt{s_{NN}}$ can be calculated by using the equation (1.2.4a): $\sqrt{s_{NN}}$ =3.16 Tev.

APPENDIX B

Here we discus Legendre transform [11]. Let us consider function $f(x)$ represented on the figure B1. We consider additional function $y = px$, where p is chosen accidentally.
Now, let us introduce new function $F(x, p)$ defined as [11]

$$F(x, p) = px - f(x)\ . \qquad \text{(B.1)}$$

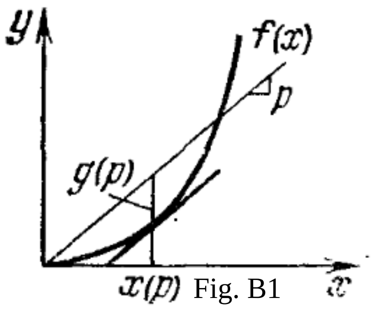

Fig. B1

It is important to emphasize that $\frac{d^2 f}{dx^2}$ is implied to be positively defined.

Maximum of the $F(x, p)$ over x corresponds to maximal distance between functions $f(x)$ and $y = px$ and can be obtained from the equation [10]:

$$\frac{\partial F}{\partial x} = 0, \qquad \text{(B.2)}$$

we obtain relation $\frac{df}{dx} = p$, which defines function $x(p)$. Therefore, for each point p we chose that very x for which relation $\frac{df}{dx} = p$ is fulfilled.

APPENDIX C

Let us prove the equation:

$$\frac{1}{N!}\sum_{\{n_\alpha\}} \frac{N!}{\prod_\alpha n_\alpha!} \prod_\alpha e^{-\frac{1}{T}\varepsilon_\alpha n_\alpha}\ \delta\left(\sum_\alpha n_\alpha - N\right) = \frac{1}{N!}\left(\sum_\alpha e^{-\frac{1}{T}\varepsilon_\alpha}\right)^N . \qquad \text{(C.1)}$$

We apply the method of mathematical induction. For N=1 we have:

$$\sum_{\{n_\alpha\}} \frac{1}{\prod_\alpha n_\alpha!} \prod_\alpha e^{-\frac{1}{T}\varepsilon_\alpha n_\alpha}\ \delta\left(\sum_\alpha n_\alpha - 1\right)$$

$$= \sum_{n_1=0}^{1} \frac{1}{n_1!} e^{-\frac{1}{T}\varepsilon_1 n_1} \sum_{n_2=0}^{1} \frac{1}{n_2!} e^{-\frac{1}{T}\varepsilon_2 n_2} \sum_{n_3=0}^{1} \frac{1}{n_3!} e^{-\frac{1}{T}\varepsilon_3 n_3} \ldots\ldots \delta\left(\sum_\alpha n_\alpha - 1\right) = \sum_\alpha e^{-\frac{1}{T}\varepsilon_\alpha}$$

$$= \frac{1}{1!}\left(\sum_\alpha e^{-\frac{1}{T}\varepsilon_\alpha}\right)^1. \qquad \text{(C.2)}$$

Let us suppose that (C.1) is right for N particles. We need to prove that (C.1) will be correct for N+1 particle:

$$(\sum_\alpha e^{-\frac{1}{T}\varepsilon_\alpha})^{N+1} = (\sum_\alpha e^{-\frac{1}{T}\varepsilon_\alpha})^{N} \times \left(\sum_{\alpha'} e^{-\frac{1}{T}\varepsilon_{\alpha'}}\right)$$

$$= \sum_{\{n_\alpha\}} \frac{N!}{\prod_\alpha n_\alpha!} \prod_\alpha e^{-\frac{1}{T}\varepsilon_\alpha n_\alpha}\ \delta(\sum_\alpha n_\alpha - N) \times \left(\sum_{\alpha'} \frac{n_{\alpha'}+1}{n_{\alpha'}+1} e^{-\frac{1}{T}\varepsilon_{\alpha'}}\right)$$

$$= \sum_{\alpha'} \sum_{n_1}^{N} \frac{1}{n_1!} e^{-\frac{1}{T}\varepsilon_1 n_1} \sum_{n_2}^{N} \frac{1}{n_2!} e^{-\frac{1}{T}\varepsilon_2 n_2} \ldots \sum_{n_{\alpha'}}^{N} \frac{n_{\alpha'}+1}{n_{\alpha'}!(n_{\alpha'}+1)} e^{-\frac{1}{T}\varepsilon_{\alpha'}(n_{\alpha'}+1)} \delta(\sum_\alpha n_\alpha - N)$$

$$= \sum_{\alpha'} \sum_{n_1}^{N} \frac{1}{n_1!} e^{-\frac{1}{T}\varepsilon_1 n_1} \sum_{n_2}^{N} \frac{1}{n_2!} e^{-\frac{1}{T}\varepsilon_2 n_2} \ldots \sum_{n_{\alpha'}=1}^{N+1} \frac{n_{\alpha'}}{n_{\alpha'}!} e^{-\frac{1}{T}\varepsilon_{\alpha'}(n_{\alpha'}+1)} \delta(\sum_{\alpha\neq\alpha} n_\alpha - (N+1-n_{\alpha'}))$$

$$= \sum_{\alpha'} \sum_{\{n_\alpha\}_{\alpha\neq\alpha}} \frac{N!}{\prod_\alpha n_\alpha!} \prod_{\alpha\neq\alpha} e^{-\frac{1}{T}\varepsilon_\alpha n_\alpha}\ \sum_{n_{\alpha'}=1}^{N+1} \frac{n_{\alpha'}}{n_{\alpha'}!} e^{-\frac{1}{T}\varepsilon_{\alpha'}(n_{\alpha'}+1)} \delta(\sum_{\alpha\neq\alpha} n_\alpha - (N+1-n_{\alpha'}))$$

$$= \sum_{\{n_\alpha\}} \frac{N!}{\prod_\alpha n_\alpha!} \prod_\alpha e^{-\frac{1}{T}\varepsilon_\alpha n_\alpha}\ \delta(\sum_\alpha n_\alpha - (N+1))\ [\sum_{\alpha'} n_{\alpha'}]$$

$$= \sum_{\{n_\alpha\}} \frac{(N+1)!}{\prod_\alpha n_\alpha!} \prod_\alpha e^{-\frac{1}{T}\varepsilon_\alpha n_\alpha}\ \delta(\sum_\alpha n_\alpha - (N+1))\,. \qquad \text{(C.3)}$$

Therefore, we have

$$\frac{1}{(N+1)!} (\sum_\alpha e^{-\frac{1}{T}\varepsilon_\alpha})^{(N+1)} = \frac{1}{(N+1)!} \sum_{\{n_\alpha\}} \frac{(N+1)!}{\prod_\alpha n_\alpha!} \prod_\alpha e^{-\frac{1}{T}\varepsilon_\alpha n_\alpha}\ \delta(\sum_\alpha n_\alpha - (N+1))\ . \qquad \text{(C.4)}$$

APPENDIX D

Here we prove equation: $d^3p = Ep_T dp_T dy d\varphi$, where $d^3p = dp_x dp_y dp_z$. The transformation from one coordinate system to another one can be easily obtained with using Jacobian determinant. Jacobian matrix has components:

$$J = \begin{pmatrix} \frac{\partial p_x}{\partial p_T} & \frac{\partial p_x}{\partial y} & \frac{\partial p_x}{\partial \varphi} \\ \frac{\partial p_y}{\partial p_T} & \frac{\partial p_y}{\partial y} & \frac{\partial p_y}{\partial \varphi} \\ \frac{\partial p_z}{\partial p_T} & \frac{\partial p_z}{\partial y} & \frac{\partial p_z}{\partial \varphi} \end{pmatrix} = \begin{pmatrix} \cos\varphi & 0 & -p_T \sin\varphi \\ \sin\varphi & 0 & p_T\cos\varphi \\ \frac{p_T}{m_T}\sinh y & m_T\cosh y & 0 \end{pmatrix}. \qquad \text{(D.1)}$$

From (D.1) we obtain determinant of this matrix:

$$|\text{Det J}| = \cos\varphi p_T \cos\varphi\, m_T \cosh y + p_T \sin\varphi \sin\varphi m_T \cosh y = p_T m_T \cosh y. \qquad \text{(D.2)}$$

Therefore, we obtain equation:

$$dp_x dp_y dp_z = |\text{Det J}| dp_T dy d\varphi = p_T m_T \cosh y dp_T dy d\varphi = E p_T dp_T dy d\varphi. \qquad \text{(D.3)}$$

Let us prove that $\frac{d^3p}{E}$ do not differ from the frame of reference [3]. The Lorentz for the four-momentum can be written as

$$E = \frac{E' + \beta p'_z}{\sqrt{1-\beta^2}}, \qquad p_z = \frac{p'_z + \beta E'}{\sqrt{1-\beta^2}}, \qquad p_x = p'_x, \quad p_y = p'_y \qquad \text{(D.4)}$$

Therefore, we can write:

$$\frac{d^3p}{E} = \frac{dp_x dp_y dp_z}{E} = \frac{\gamma(dp'_z+\beta dE')dp'_x dp'_y}{\gamma(E'+\beta p'_z)} = \frac{\gamma dp'_z(1+\frac{\beta dE'}{dp'_z})dp'_x dp'_y}{\gamma E'(1+\frac{\beta p'_z}{E'})}, \qquad \text{(D.5)}$$

Using relation $EdE = p_z dp_z$ [3], we obtain:

$$\frac{d^3p}{E} = \frac{d^3p'}{E'}. \qquad \text{(D.6)}$$

Hence this infinitesimal element $\frac{d^3p}{E}$ is relativistic invariant.

APPENDIX E

Here we solve integral:

$$I = \int_0^\infty e^{-\frac{1}{T}(\varepsilon_p-\mu)} p^2 dp = e^{\frac{1}{T}\mu} \int_0^\infty e^{-\frac{1}{T}\sqrt{p^2+m^2}} p^2 dp, \qquad \text{(E.1)}$$

for that we express p via new variable:

$$p = m \sinh t\ . \qquad \text{(E.2)}$$

Substituting (E.2) into (E.1), we obtain

$$e^{\frac{1}{T}\mu} \int_0^\infty e^{-\frac{1}{T}\sqrt{p^2+m^2}} p^2 dp = e^{\frac{1}{T}\mu} \int_0^\infty e^{-\frac{1}{T}m\cosh t} (m\sinh t)^2\, d(m\sinh t)$$

$$= e^{\frac{1}{T}\mu} \int_0^\infty e^{-\frac{1}{T}m\cosh t} (\cosh^2 t - 1) m^3 \cosh t\, dt$$

$$= e^{\frac{1}{T}\mu} m^3 \left[\int_0^\infty e^{-\frac{1}{T}m\cosh t} \cosh^3 t\, dt - \int_0^\infty e^{-\frac{1}{T}m\cosh t} \cosh t\, dt\right]$$

$$= e^{\beta\mu} m \frac{\partial^2}{\partial\beta^2} K_1(m\beta) - e^{\beta\mu} m^3 K_1(m\beta) = e^{\beta\mu} m \left[m^2 \frac{\partial^2}{\partial(m\beta)^2} K_1(m\beta) - m^2 K_1(m\beta)\right], \qquad \text{(E.3)}$$

where $K_1(m\beta) = \int_0^\infty e^{-\frac{1}{T}m\cosh t} \cosh t\, dt$ is modified Bessel function in integral representation [20]. The modified Bessel function satisfies the equation [20]:

$$\frac{d^2K_n}{dx^2} + \frac{1}{x}\frac{dK_n}{dx} - (1+\frac{n^2}{x^2})K_n\ . \qquad \text{(E.4)}$$

Therefore, we obtain

$$\frac{d^2K_n}{dx^2} = -\frac{1}{x}\frac{dK_n}{dx} + (1+\frac{n^2}{x^2})K_n\ . \qquad \text{(E.4a)}$$

Modified Bessel function K_n satisfies recurrence relations [20]:

$$\frac{\partial}{\partial x}(e^{i\pi n}K_n) = e^{i\pi(n-1)}K_{n-1} - \frac{n}{x}e^{i\pi n}K_n, \qquad \text{(E.5)}$$

$$e^{i\pi(n+1)}K_{n+1} = e^{i\pi(n-1)}K_{n-1} - \frac{2}{x}e^{i\pi n}K_n\ . \qquad \text{(E.6)}$$

Substituting (E.5) and (E.4a) into (E.3), we obtain eventual relation for the integral I:

$$I = e^{\beta\mu} m^3 \left[-\frac{1}{m\beta}\frac{dK_1}{d(m\beta)} + (1 + \frac{1^2}{(m\beta)^2})K_1 - K_1(m\beta) \right] = e^{\beta\mu} m^3 \left[\frac{1}{m\beta}(K_0 + \frac{2}{m\beta}K_1) \right]$$
$$= e^{\beta\mu}\frac{m^2}{\beta}K_2. \qquad \text{(E.7)}$$

APPENDIX F

Here we prove relation (5.2.24) [4]. Let us write right-hand side of the equation:

$$e^{-\frac{1}{T}\varepsilon_{\alpha p}} \sum_{n_{\alpha p}=0}^{\infty} \frac{1}{n_{\alpha p}!} e^{-\frac{1}{T}\varepsilon_{\alpha p} n_{\alpha p}} \delta(\sum_{\alpha' \in M_\pm} \sum_{p'} |s_{\alpha'}| n_{\alpha' p'} - (S_\pm - |s_\alpha|))$$

$$= \sum_{n_{\alpha p}=0}^{\infty} \frac{1}{n_{\alpha p}!} e^{-\frac{1}{T}\varepsilon_{\alpha p}(n_{\alpha p}+1)} \delta(\sum_{\alpha' \in M_\pm} \sum_{p'} |s_{\alpha'}| n_{\alpha' p'} - (S_\pm - |s_\alpha|))$$

$$= \sum_{n_{\alpha p}=1}^{\infty} \frac{1}{(n_{\alpha p}-1)!} e^{-\frac{1}{T}\varepsilon_{\alpha p} n_{\alpha p}} \delta(\sum_{\alpha' \in M_\pm} \sum_{p' \neq p} |s_{\alpha'}| n_{\alpha' p'} + |s_\alpha|(n_{\alpha p} - 1) - (S_\pm - |s_\alpha|))$$

$$= \sum_{n_{\alpha p}=0}^{\infty} \frac{n_{\alpha p}}{n_{\alpha p}!} e^{-\frac{1}{T}\varepsilon_{\alpha p} n_{\alpha p}} \delta(\sum_{\alpha' \in M_\pm} \sum_{p'} |s_{\alpha'}| n_{\alpha' p'} - S_\pm). \qquad \text{(F.1)}$$

Let us also prove equation (5.2.37) [4]:

$$\sum_{n_{\alpha' p'}=0}^{K_{\alpha'}} n_{\alpha' p'} e^{-\frac{1}{T} n_{\alpha p}\varepsilon_{\alpha p}} \delta(\sum_{\alpha \in M_\pm} \sum_p |s_\alpha| n_{\alpha p} - S_\pm)$$

$$= \sum_{l=1}^{\left[\frac{S_\pm}{|s_{\alpha'}|}\right]} y_{\alpha' l} e^{-\frac{1}{T} l \varepsilon_{\alpha' p'}} \sum_{n_{\alpha' p'=0}}^{min\left\{\left[\frac{S_\pm}{|s_{\alpha'}|}\right]-l,\, K_{\alpha'}\right\}} e^{-\frac{1}{T} n_{\alpha' p'}\varepsilon_{\alpha' p'}} \delta(\sum_{\alpha \in M_\pm} \sum_p |s_\alpha| n_{\alpha p} - (S_\pm - l|s_{\alpha'}|)), \qquad \text{(F.2)}$$

where K_α defines maximal occupation of the state: for the Fermi-Dirac statistics $K_\alpha = 1$, for the Bose-Einstein statistics $K_\alpha = \infty$ [19].

Let us now focus on Bose-Einstein statistics:

$$\sum_{l=1}^{\left[\frac{S_\pm}{|s_{\alpha'}|}\right]} e^{-\frac{1}{T} l \varepsilon_{\alpha' p}} \sum_{n_{\alpha' p'=0}}^{\infty} e^{-\frac{1}{T} n_{\alpha' p'}\varepsilon_{\alpha' p'}} \delta(\sum_{\alpha \in M_\pm} \sum_p |s_\alpha| n_{\alpha p} - (S_\pm - l|s_{\alpha'}|))$$

$$= \sum_{l=1}^{\left[\frac{S_\pm}{|s_{\alpha'}|}\right]} \sum_{n_{\alpha' p'=0}}^{\infty} e^{-\frac{1}{T}(n_{\alpha' p'}+l)\varepsilon_{\alpha' p'}} \delta(\sum_{\alpha \in M_\pm} \sum_p |s_\alpha| n_{\alpha p} - (S_\pm - l|s_{\alpha'}|))$$

$$= \sum_{l=1}^{\left[\frac{S_\pm}{|s_{\alpha'}|}\right]} \sum_{n_{\alpha p''}=l}^{\infty} e^{-\frac{1}{T} n_{\alpha' p'}\varepsilon_{\alpha' p'}} \delta(\sum_{\alpha \in M_\pm, \alpha \neq \alpha'} \sum_{p \neq p'} |s_\alpha| n_{\alpha p} + |s_{\alpha'}|(n_{\alpha' p'} - l) - (S_\pm - l|s_{\alpha'}|)) =$$
$$\sum_{l=1}^{\left[\frac{S_\pm}{|s_{\alpha'}|}\right]} \sum_{n_{\alpha' p'}=l}^{\infty} e^{-\frac{1}{T} n_{\alpha' p'}\varepsilon_{\alpha' p'}} \delta(\sum_{\alpha \in M_\pm} \sum_p |s_\alpha| n_{\alpha p} - S_\pm). \qquad \text{(F.3)}$$

Substituting $\Theta(n_{\alpha p} - l)$ into (F.3), we obtain

$$\sum_{l=1}^{\left[\frac{S_\pm}{|s_{\alpha'}|}\right]} \sum_{n_{\alpha' p'}=0}^{\infty} \Theta(n_{\alpha' p'} - l) e^{-\frac{1}{T} n_{\alpha' p'}\varepsilon_{\alpha' p'}} \delta(\sum_{\alpha \in M_\pm} \sum_p |s_\alpha| n_{\alpha p} - S_\pm)$$

$$= \sum_{n_{\alpha' p'}=0}^{\infty} \sum_{l=1}^{\left[\frac{S_\pm}{|s_{\alpha'}|}\right]} \Theta(n_{\alpha' p'} - l) e^{-\frac{1}{T} n_{\alpha' p'}\varepsilon_{\alpha' p'}} \delta(\sum_{\alpha \in M_\pm} \sum_p |s_\alpha| n_{\alpha p} - S_\pm)$$

$$= \sum_{n_{\alpha' p'}=0}^{\infty} n_{\alpha' p'} e^{-\frac{1}{T} n_{\alpha' p'}\varepsilon_{\alpha' p'}} \delta(\sum_{\alpha \in M_\pm} \sum_p |s_\alpha| n_{\alpha p} - S_\pm)\,. \qquad \text{(F.4)}$$

For the Fermi-Dirac statistics $y_{\alpha l} = (-1)^{l+1}$. Writing all components of the sum, we obtain

$$\sum_{l=1}^{\left[\frac{S_\pm}{|s_{\alpha'}|}\right]}(-1)^{l+1}e^{-\frac{1}{T}l\varepsilon_{\alpha' p'}}\sum_{n_{\alpha' p'}=0}^{min\left\{\left[\frac{S_\pm}{|s_{\alpha'}|}\right]-l,1\right\}} e^{-\frac{1}{T}n_{\alpha' p'}\varepsilon_{\alpha' p'}}\delta(\sum_{\alpha\in M_\pm}\sum_p |s_\alpha| n_{\alpha p} - (S_\pm - l|s_{\alpha'}|))$$

$$= e^{-\frac{1}{T}\varepsilon_{\alpha' p'}}\delta(\sum_{\alpha\in M_\pm}\sum_p |s_\alpha| n_{\alpha p} - (S_\pm - |s_{\alpha'}|))$$

$$= \sum_{n_{\alpha' p'}=0}^{1} n_{\alpha' p'} e^{-\frac{1}{T}n_{\alpha' p'}\varepsilon_{\alpha' p'}}\delta(\sum_{\alpha\in M_\pm}\sum_p |s_\alpha| n_{\alpha p} - S_\pm). \quad \text{(F.5)}$$

APPENDIX G

Here we consider Z_0 (5.2.35) $-$component of the partition function corresponding to species $\alpha \in M_0$ [6]:

$$Z_0 = \prod_{\alpha\in M_0}\sum_{n_\alpha}\left[\sum_{\{N_{\alpha l}\}'\alpha\in M_0}\delta\left(\sum_{l=1}^{n_\alpha} lN_{\alpha l} - n_\alpha\right)\prod_{l=1}^{n_\alpha}\frac{(Z_\alpha^l)^{N_{\alpha l}}}{l^{N_{\alpha l}}N_{\alpha l}!}\right]. \quad \text{(G.1)}$$

Let us obtain this relation for the particular case. First of all we write canonical partition functions for quantum statistics in general form. Substituting (3.1.22) into equation (5.2.39), we obtain

$$Z_0 = \prod_{\alpha\in M_0}\sum_{n_\alpha}[Z_{n_\alpha}], \quad \text{(G.2)}$$

where

$$Z_{n_\alpha} = \frac{1}{n_\alpha}\sum_{l=1}^{n_\alpha} Z_\alpha^l Z_{n_\alpha - l},$$

$$Z_\alpha^l = y_{\alpha l}\sum_{p'} e^{-\frac{1}{T}l\varepsilon_{\alpha p}}.$$

Compering (G.1) and (G.2), it becomes obvious that all we need is to prove:

$$\sum_{\{N_{\alpha l}\}'\alpha\in M_0}\delta\left(\sum_{l=1}^{n_\alpha} lN_{\alpha l} - n_\alpha\right)\prod_{l=1}^{n_\alpha}\frac{(Z_\alpha^l)^{N_{\alpha l}}}{l^{N_{\alpha l}}N_{\alpha l}!} = Z_{n_\alpha}. \quad \text{(G.3)}$$

Let us put $n_\alpha = 4$. Therefore, we obtain

$$Z_4 = \frac{1}{4}\sum_{l=1}^{4} Z_\alpha^l Z_{4-l} = \frac{1}{4}[Z_\alpha^1 Z_3 + Z_\alpha^2 Z_2 + Z_\alpha^3 Z_1 + Z_\alpha^4 Z_0], \quad \text{(G.4)}$$

where $Z_0 = 1$. We have:

$$Z_4 = \frac{1}{4}\left\{\frac{Z_\alpha^1}{3}(Z_\alpha^1 Z_2 + Z_\alpha^2 Z_1 + Z_\alpha^3) + \frac{Z_\alpha^2}{2}(Z_\alpha^1 Z_1 + Z_\alpha^2) + Z_\alpha^3 Z_\alpha^1 Z_0 + Z_\alpha^4\right\}$$

$$= \frac{1}{4}\left\{\frac{1}{3}\left(\frac{\left((Z_\alpha^1)^4 + Z_\alpha^2 (Z_\alpha^1)^2\right)}{2} + Z_\alpha^2 (Z_\alpha^1)^2 + Z_\alpha^3 Z_\alpha^1\right) + \frac{1}{2}(Z_\alpha^2 (Z_\alpha^1)^2 + (Z_\alpha^2)^2) + Z_\alpha^3 Z_\alpha^1 + Z_\alpha^4\right\}$$

$$= \frac{(Z_\alpha^1)^4 (Z_\alpha^2)^0 (Z_\alpha^3)^0 (Z_\alpha^4)^0}{24} + \frac{(Z_\alpha^1)^2 (Z_\alpha^2)^1 (Z_\alpha^3)^0 (Z_\alpha^4)^0}{4} + \frac{(Z_\alpha^1)^1 (Z_\alpha^2)^0 (Z_\alpha^3)^1 (Z_\alpha^4)^0}{3} + \frac{(Z_\alpha^1)^0 (Z_\alpha^2)^2 (Z_\alpha^3)^0 (Z_\alpha^4)^0}{8}$$

$$+ \frac{(Z_\alpha^1)^0 (Z_\alpha^2)^0 (Z_\alpha^3)^0 (Z_\alpha^4)^1}{4}. \quad \text{(G.5)}$$

Let us compare coefficients in (G.3) with obtained

$$= \frac{(z_\alpha^1)^4 (z_\alpha^2)^0 (z_\alpha^3)^0 (z_\alpha^4)^0}{1^4 4!} + \frac{(z_\alpha^1)^2 (z_\alpha^2)^1 (z_\alpha^3)^0 (z_\alpha^4)^0}{1^2 2! 2^1 1!} + \frac{(z_\alpha^1)^1 (z_\alpha^2)^0 (z_\alpha^3)^1 (z_\alpha^4)^0}{3^1 1!} + \frac{(z_\alpha^1)^0 (z_\alpha^2)^2 (z_\alpha^3)^0 (z_\alpha^4)^0}{2! 2^2}$$

$$+ \frac{(z_\alpha^1)^0 (z_\alpha^2)^0 (z_\alpha^3)^0 (z_\alpha^4)^1}{4^1 1!}. \qquad \text{(G.6)}$$

All coefficients match. Also, we need to check if all possible combinations of $N_{\alpha l}$ are presented in (G.6): $n_\alpha = 4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1$. Therefore, possible combinations of $N_{\alpha l}$: {0,0,0,1}, {1,0,1,0}, {0,2,0,0}, {2,1,0,0}, {4,0,0,0}.

1) $\frac{(z_\alpha^1)^4 (z_\alpha^2)^0 (z_\alpha^3)^0 (z_\alpha^4)^0}{1^4 4!}$ correspond to case when four states are occupied and contain one particle $(1 + 1 + 1 + 1)$;
2) $\frac{(z_\alpha^1)^2 (z_\alpha^2)^1 (z_\alpha^3)^0 (z_\alpha^4)^0}{1^2 2! 2^1 1!}$ correspond to case when two states are occupied by single particles and one state occupied by two particles $(2 + 1 + 1)$;
3) $\frac{(z_\alpha^1)^1 (z_\alpha^2)^0 (z_\alpha^3)^1 (z_\alpha^4)^0}{3^1 1!}$ correspond to case when one state occupied by one particle and one state occupied by three particles $(3 + 1)$;
4) $\frac{(z_\alpha^1)^0 (z_\alpha^2)^0 (z_\alpha^3)^0 (z_\alpha^4)^1}{4^1 1!}$ correspond to case when all four particles are gathered in one state.

Hence all possible combinations are figured in (G.6).

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