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FINAL REPORT ON THE START PROGRAMME

Rotation-Invariants and Constraints for Leptonic Decay of Charmonium

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1 Abstract

Exploring dilepton decay of charmonium is a way to study polarisation of charmonium. Angular parameters or coefficients which are included in angular distribution of dilepton are frame-dependent, that is why finding any relations for them is significant for analysis. Another important task is to set relations for these parameters and initial charmonium state which is described with a hadron tensor. Besides, properties of the tensor bound values of the angular coefficients and rotation-invariants of the distribution. Thus, the approach studied and stated in this report can be very useful for charmonium research.

2 Introduction

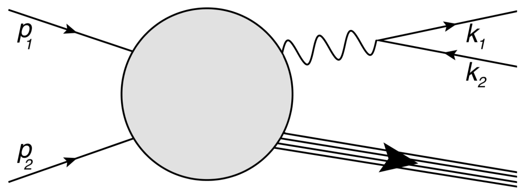
Studying and description of process of quarkonium production and subsequent dilepton decay is a fine task for QCD testing. Angular distribution of dilepton decay of quarkonium is parameterised with a set of angular coefficients which are dependant on definition of coordinate system axes. One of the parameters for charmonium decay was experimentally measured by PHENIX collaboration [1], for example. Because the parameters are frame-dependent, it is useful for analysis to have quantities which could relate them with each other, i.e. some invariant quantities.

Firstly, we will derive a form of the angular distribution without usage of any factorisation and hadronisation approach but on the basis of quite general ideas. Also, we will try to express a spacial part of a hadron tensor (which describes an initial quarkonium state before decay) in terms of the observable angular coefficients. Exploring the existence and properties of the invariants will be associated with a question of some symmetry in angular distribution and a frame that could provide such symmetry. Further, we will note bounds which are imposed by properties of the hadron tensor.

The approach for charmonium research studied by me is a helpful and powerful tool for analysis of both experimental data and theoretical predictions.

3 Leptonic decay of charmonium

We will start with the most general ideas of description of charmonium production and then find connection of charmonium polarisation and angular distribuition of leptons produced during charmonium decay.



On the diagram of our process in a one-photon approximation a grey blob refers to a hadron tensor $W_{\mu\nu}$ in a final squared amplitude and an external leptonic line refers to a lepton tensor $L_{\mu\nu}$.

And a cross section of the process will be proportional to a contraction of them:

$$d\sigma \sim L_{\mu\nu} W^{\mu\nu}.$$

The hadron tensor should follow conditions of symmetry, parity, gauge invariance and positivity. According to them, the hadron tensor $W_{\mu\nu}$ can be written as a combination of structural functions W_i:

$$W_{\mu\nu} = W_1 \widetilde{g}_{\mu\nu} + W_2 \widetilde{P}_\mu \widetilde{P}_\nu - \frac{W_3}{2}\left(\widetilde{P}_\mu \widetilde{p}_\nu + \widetilde{p}_\mu \widetilde{P}_\nu\right) + W_4 \widetilde{p}_\mu \widetilde{p}_\nu,$$

$$\widetilde{g}_{\mu\nu} = g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}, \qquad \widetilde{P}_\mu = \widetilde{g}_{\mu\nu} \frac{P^\nu}{\sqrt{s}}, \qquad \widetilde{p}_\mu = \widetilde{g}_{\mu\nu} \frac{p^\nu}{\sqrt{s}},$$

$$P = p_1 + p_2, \qquad p = p_1 - p_2, \qquad q = k_1 + k_2, \qquad k = k_1 - k_2.$$

But the explicit form of the tensor won't be used. Instead, we consider the process in a rest frame of charmonium, thus, a sum of leptons' momenta is $q^\mu = k_1^\mu + k_2^\mu = (q_0, 0, 0, 0)$. Therefore, we can confine ourselves to exploration of the spacial part of the $W_{\mu\nu}$ due to the lepton current conservation $q_\mu W^{\mu\nu} = W^{\mu\nu} q_\mu = 0$. Also, we will use the helicity amplitudes for hadron tensor which are defined as

$$W_{\sigma,\sigma'} = \varepsilon_\sigma^\mu(q)\, W_{\mu\nu}\, \varepsilon_{\sigma'}^{*\nu}(q) \tag{1}$$

where $\varepsilon_\sigma^\mu(q)$ is polarisation four-vectors of a virtual photon in a coordinate system where $\varepsilon_{(0)}^\mu(q) = Z^\mu$ and $\varepsilon_{(\pm1)}^\mu(q) = \frac{1}{\sqrt{2}}(\mp X^\mu - iY^\mu)$ [2]. The four-vectors X^μ, Y^μ, Z^μ are four-dimensional extension of a three-dimensional cartesian coordinate system vectors with time components being equal to zero.

Using (1) and definition of polarisation vectors we can get the tensor $W_{\mu\nu}$ in a form

$$W_{\mu\nu} = \widetilde{g}_{\mu\nu}\left(W_T + W_{\Delta\Delta}\right) - 2X_\mu X_\nu W_{\Delta\Delta} + Z_\mu Z_\nu \left(W_L - W_T - W_{\Delta\Delta}\right) - \left(X_\mu Z_\nu + Z_\mu X_\nu\right) W_\Delta$$

where independent helicity amplitudes are left only, they are redenoted according to

$$W_T = W_{1,1}, \qquad W_L = W_{0,0}, \qquad W_\Delta = \frac{W_{1,0} + W_{0,1}}{\sqrt{2}}, \qquad W_{\Delta\Delta} = W_{1,-1}.$$

Contraction with the $L_{\mu\nu}$ tensor and explicit form of k^μ-components in a spherical system connected with the cartesian system $(X^\mu,\ Y^\mu,\ Z^\mu)$ give angular dependence of the cross section in one of the convenient alternatives of W components (we will further note with W the spacial part of the hadron tensor):

$$\frac{d\sigma}{d\Omega} \sim L_{\mu\nu} W^{\mu\nu} = W_T\left(1 + \cos^2\theta\right) + W_L\left(1 - \cos^2\theta\right) + W_\Delta \sin 2\theta \cos\varphi + W_{\Delta\Delta} \sin^2\theta \cos 2\varphi =$$

$$= (W_{1,1} + W_{0,0}) + (W_{1,1} - W_{0,0})\cos^2\theta + \frac{(W_{1,0} + W_{0,1})}{\sqrt{2}} \sin 2\theta \cos\varphi + W_{1,-1} \sin^2\theta \cos 2\varphi.$$

Now we can obtain connection of J/ψ polarisation with the $W_{\mu\nu}$ tensor. If one considers unpolarised charmonium state as a superposition of polarised states with values of angular momentum projections $J_z = 0, \pm 1$ [3]:

$$|J/\psi\rangle = a_1\,|1\rangle + a_0\,|0\rangle + a_{-1}\,|{-1}\rangle\,, \qquad \mathcal{N} = |a_1|^2 + |a_0|^2 + |a_{-1}|^2, \tag{2}$$

then the helicity amplitudes $W_{\sigma,\sigma'}$ are proportional to $a_\sigma a^*_{\sigma'}$ with some coefficients which won't be specified here as they are going to be reduced in expressions of observables. The coefficients in the contraction $L_{\mu\nu}W^{\mu\nu}$ can be written as

$$W_{1,1} - W_{0,0} \sim \frac{1}{2}\left(\mathcal{N} - 3|a_0|^2\right), \qquad W_{1,1} + W_{0,0} \sim \frac{1}{2}\left(\mathcal{N} + |a_0|^2\right),$$

$$W_{1,0} + W_{0,1} \sim 2\,\mathrm{Re}\left(a_0 a_1^*\right), \qquad W_{1,-1} \sim \mathrm{Re}\left(a_1 a_{-1}^*\right),$$

and we can now write the final angular distribution in terms of angles of a chosen frame and observable coefficients λ, μ and ν:

$$\frac{d\sigma}{d\Omega} \sim 1 + \lambda\cos^2\theta + \mu\sin 2\theta\cos\varphi + \nu\sin^2\theta\cos 2\varphi, \tag{3}$$

$$\lambda = \frac{W_{1,1} - W_{0,0}}{W_{1,1} + W_{0,0}} = \frac{\mathcal{N} - 3|a_0|^2}{\mathcal{N} + |a_0|^2}, \qquad \nu = \frac{W_{1,-1}}{W_{1,1} + W_{0,0}} = \frac{2\,\mathrm{Re}\left(a_1 a_{-1}^*\right)}{\mathcal{N} + |a_0|^2},$$
$$\mu = \frac{W_{1,0} + W_{0,1}}{\sqrt{2}\,(W_{1,1} + W_{0,0})} = \frac{2\sqrt{2}\,\mathrm{Re}\left(a_0 a_1^*\right)}{\mathcal{N} + |a_0|^2}.$$

In order to get rid of the uncertainty in definition of the angular distribution we will integrate the cross section of the lepton production over the solid angle:

$$\frac{1}{\sigma}\frac{d\sigma}{d\Omega} = \frac{3}{4\pi}\frac{1}{3+\lambda}\left(1 + \lambda\cos^2\theta + \mu\sin 2\theta\cos\varphi + \nu\sin^2\theta\cos 2\varphi\right).$$

Except the last expression, we can find the explicit form of the W as well. It can be useful to express it in terms of the helicity amplitudes $W_{\sigma,\sigma'}$ (if necessary, normalised matrix can be obtained with the help of the W normalisation condition $\mathrm{Tr}\,W = 1$):

$$W = \begin{pmatrix} W_T - W_{\Delta\Delta} & 0 & -W_\Delta \\ 0 & W_T + W_{\Delta\Delta} & 0 \\ -W_\Delta & 0 & W_L \end{pmatrix} =$$
$$= \begin{pmatrix} W_{1,1} - W_{1,-1} & 0 & -(W_{1,0} + W_{0,1})/\sqrt{2} \\ 0 & W_{1,1} + W_{1,-1} & 0 \\ -(W_{1,0} + W_{0,1})/\sqrt{2} & 0 & W_{0,0} \end{pmatrix}.$$

And, of course, the same matrix (normalised this time) which components are represented as functions of the angular coefficients is

$$W = \frac{2}{3+\lambda}\begin{pmatrix} \frac{1+\lambda-2\nu}{2} & 0 & -\mu \\ 0 & \frac{1+\lambda+2\nu}{2} & 0 \\ -\mu & 0 & \frac{1-\lambda}{2} \end{pmatrix}.$$

The coefficients can be experimentally measured and calculated that's why the W components can be considered as found. It is only enough to remember what the coefficients $a_{0,\pm1}$ mean in the sum (2): we are able to multiply both numerators and denominators of the λ, μ, ν expressions with necessary factors to obtain them as combinations of (un)polarised cross sections of charmonium. For instance,

$$\lambda = \frac{d\sigma - 3\,d\sigma_{0,0}}{d\sigma + d\sigma_{0,0}} \equiv \frac{d\sigma - 3\,d\sigma_{\mathrm{L}}}{d\sigma + d\sigma_{\mathrm{L}}} \equiv \frac{d\sigma_{\mathrm{T}} - 2\,d\sigma_{\mathrm{L}}}{d\sigma_{\mathrm{T}} + 2\,d\sigma_{\mathrm{L}}}.$$

We haven't specified any frame for derivation of the angular distribution. And values of the angular coefficients differ depending on the frame. That's why the next stage for us will be looking for some invariants that can mathematically connect different frames.

4 Invariants for charmonium decay

Invariant of reduced distribution

We can see that in the distribution (3) two of four terms don't depend on φ. Therefore, one can suppose that there is a frame where the distribution includes dependence on θ only:

$$\frac{d\sigma}{d\Omega} \sim 1 + \lambda_0 \cos^2\theta'$$

where θ' is an polar angle in the new frame. The connection of this angle and angles in an arbitrary frame is

$$\cos\theta' = \sin\theta \sin\theta_0 \cos\varphi + \cos\theta \cos\theta_0$$

where θ is a polar angle in the arbitrary frame, θ_0 is an angle between two z-axes, and φ is an azimuthal angle of z-axis direction of the new frame. If we substitute $\cos\theta'$ in the distribution and reorganise it in such way that it has similar structure as (3), the parameters λ, μ, ν will be able to be expressed in terms of λ_0 and θ_0 in the following way:

$$\lambda = \lambda_0 \frac{1 - 3\frac{\sin^2\theta_0}{2}}{1 + \lambda_0\frac{\sin^2\theta_0}{2}}, \qquad \mu = \lambda_0 \frac{\sin\theta_0 \cos\theta_0}{1 + \lambda_0\frac{\sin^2\theta_0}{2}}, \qquad \nu = \lambda_0 \frac{\frac{\sin^2\theta_0}{2}}{1 + \lambda_0\frac{\sin^2\theta_0}{2}}. \tag{4}$$

Expressing λ_0 and $\sin\theta_0$, we obtain it as a combination of the parameters λ and ν in the arbitrary frame:

$$\lambda_0 = \frac{\lambda + 3\nu}{1 - \nu} = \frac{W_{1,1} - W_{0,0} + 3W_{1,-1}}{W_{1,1} + W_{0,0} - W_{1,-1}},$$

$$\sin^2\theta_0 = \frac{2\nu}{\lambda + 3\nu} = \frac{2W_{1,-1}}{W_{1,1} - W_{0,0} + 3W_{1,-1}}.$$

Thus, assuming that the frame without azimuthal dependence of the angular distribution exists, we were able to find expressions of λ_0 and the angle that geometrically connect two frames. The value of the angle θ_0 obviously would differ for different frames. But λ_0 is constant for different frames and the same kinematic parameters. That's why we can take it (if the corresponding frame exists) as the first invariant though we've derived it quite heuristically.

Frame of azimuthal symmetry

The main question of the previous part is whether the discussed frame exists. It's useful to look onto the W matrix to answer this question. In such frame the coefficients μ and ν have zero values that's why W becomes a diagonal matrix. A frame where any matrix is diagonal is a frame of its eigenvectors as basis vectors. The set of eigenvectors of W is

$$\left\{(0,1,0)\,,\;\left(\left[-\frac{\lambda-\nu}{2\mu}+\sqrt{\left(\frac{\lambda-\nu}{2\mu}\right)^2+1}\,\right],0,1\right),\;\left(\left[-\frac{\lambda-\nu}{2\mu}-\sqrt{\left(\frac{\lambda-\nu}{2\mu}\right)^2+1}\,\right],0,1\right)\right\}.$$

If we change the coordinate system of W to the one which has upper set as basis vectors and then use the expressions (4) and the definition of $\sin^2\theta_0$ — we obtain just the same diagonal form of the W as we get if we would nullify μ and ν coefficients. Thus, we apparently can conclude that in the frame of W's eigenvectors the angular distribution of charmonium decay into pair of leptons is azimuthally symmetric.

It's also important to mention that the set of normalised eigenvectors in limit of $\mu \to 0$ becomes a set of three orthonormal vectors directed along the x, y, z axes — just as it should be expected.

By the way, the y axis remains untouched while the necessary rotation of the frame is being made. Therefore, one should rotate the frame around the y axis to the angle θ_0 to obtain the azimuthal distribution.

Rotation invariants

The other way of finding invariants is to pay attention to properties of the W matrix itself. The value of its trace won't be helpful as it is equal to one. But its eigenvalues could be interesting for analysis:

$$w_1 = \frac{1+\lambda+2\nu}{3+\lambda}, \qquad w_{2,3} = \frac{1-\nu\pm\sqrt{(\lambda-\nu)^2+4\mu^2}}{3+\lambda}.$$

The w_1 value coincides with W_{22} component of the matrix which is clear because of the untouchability of the y axis during described frame rotation. Invariants are not only $w_{1,2,3}$ but any combinations of them that's why they can be used to derive another invariants which are shorter or more convenient to apply. For example, it's easy to notice an invariant, though it doesn't include μ anymore:

$$\frac{w_2+w_3}{2} = \frac{1-\nu}{3+\lambda}.$$

Moreover, the λ_0 coefficient can be expressed in terms of the $w_{1,2,3}$ invariants as it is an invariant itself:

$$\lambda_0 = \frac{2w_1-w_2-w_3}{w_2+w_3}.$$

5 Positivity constraints

Properties of objects described earlier constraint the angular coefficients and consequently invariants for charmonium decay. Bounds for diagonal elements of the W ($0 \leqslant W_{ii} \leqslant 1$) and determinant of the W matrix ($\det W > 0$) cause following constraints for the coefficients:

$$|\lambda| \leqslant 1, \qquad |2\nu| \leqslant 1 + \lambda, \qquad \mu^2 \leqslant \frac{(1+\lambda-2\nu)(1-\lambda)}{4}. \tag{5}$$

The same bound for the diagonal matrix and its unit trace

$$0 \leqslant w_{1,2,3} \leqslant 1, \qquad w_1 + w_2 + w_3 = \mathrm{Tr}\, W = 1$$

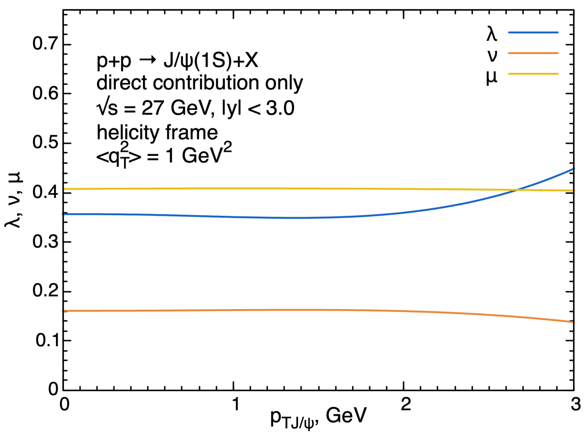
give a maximum value of these combinations of the eigenvalues:

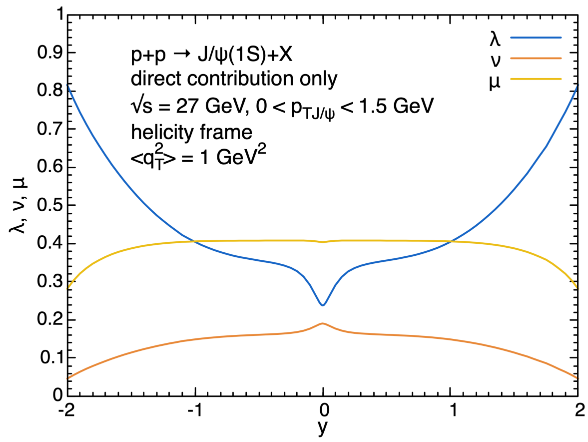
$$0 \leqslant w_1 w_2 w_3 \leqslant \frac{1}{27}, \qquad 0 \leqslant w_1 w_2 + w_1 w_3 + w_2 w_3 \leqslant \frac{1}{3} \quad \text{and} \quad 0 \leqslant \frac{w_2 + w_3}{2} \leqslant \frac{1}{2}.$$

Beside all these constraints, there are bounds for combinations of λ and ν which represent λ_0 and $\sin^2\theta_0$ that are connected with the frame of azimuthal distribution. But the constraints (5) automatically ensure realisation of $-1 \leqslant \lambda_0 \leqslant 1$ and $0 \leqslant \sin^2\theta_0 \leqslant 1$.

6 Example of calculation for NICA

An example of calculation of the λ, μ, ν coefficients for charmonium production in proton–proton collisions for NICA kinematics was made within General Parton Model (GPM) as a factorisation model and Non-relativistic QCD as a hadronisation model of charmonium. Direct contribution of J/ψ only was accounted. A GPM parameter $\langle q_T^2 \rangle$ was traditionally taken equal to 1 GeV^2. A helicity frame was used as a frame for definition of polarisation, a vector of longitudinal polarisation is directed along the charmonium three-dimensional momentum within this frame. The following plots demonstrate dependence of the coefficients on transverse momentum of charmonium and its rapidity y.





The invariants for direct charmonium production don't depend on either transverse momentum or rapidity of charmonium:

$$w_1 = 0.5 \pm (2 \cdot 10^{-11}), \qquad w_2 = 0.5 \pm (3 \cdot 10^{-4}), \qquad w_3 = (3.6 \cdot 10^{-6}) \pm (3 \cdot 10^{-4}),$$

that is apparently a peculiarity of the used models or the fact that only direct contribution was calculated within these models. As one can see, the invariants satisfy corresponding constraints and give the value of the $\lambda_0 = 1$ with good accuracy.

7 Conclusion

Studied during the practice and described here instruments can be applied to future research and calculations such as numerical calculations of the angular coefficients. Besides, this very useful tool of analysis can be used and has already been used not for quarkonium studying only but for weak decays as well [4].

8 Acknowledgments

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