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# FINAL REPORT ON THE SUMMER STUDENT PROGRAM

# *Solutions of the Yang-Baxter equation with exceptional Lie group symmetry*

**Student:**
Ivan Sergeyev, Russia
MIPT

**Supervisor:**
Prof. Alexey Petrovich Isaev

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**Abstract**

In this work, solutions of the Yang-Baxter equation with exceptional Lie group symmetries are obtained and analyzed. First, we construct the exceptional Lie algebra $\mathfrak{g}_2$ and find a solution of the Yang-Baxter equation possessing $G_2$ symmetry. The structure of the obtained solution is explained in terms of irreducible representations of $G_2$. We then perform the same steps for the exceptional Lie group $F_4$. Finally, we construct the exceptional Lie algebra $\mathfrak{e}_6$ and provide some ideas about the form of the solution with $E_6$ symmetry.

# 1 Introduction

The Yang-Baxter equation is the cornerstone of integrability of quantum theories. The equation also arises in various other fields, for example, statistical mechanics, knot theory and braid theory ([1]). In this work we investigate the class of solutions of the Yang-Baxter equation that possess simple Lie group symmetry, as it plays an important role in quantum integrability theory and 2D scattering theory.

The text is organized as follows. In Section 2 we give basic definitions related to the Yang-Baxter equation, outline its properties and provide a simple example. Then, we consider the two smallest simple Lie groups $G_2$ and $F_4$ in Sections 3 and 4. For each the respective Lie algebras are constructed and the respective solutions of the Yang-Baxter equation are found. We analyze both solutions from the point of view of representation theory and give relations between tensors that can be used to express the solution. Finally, in Section 5 we construct the basis of the exceptional Lie algebra $\mathfrak{e}_6$ and provide some ideas about the form of the solution with $E_6$ symmetry.

# 2 Yang-Baxter equation

## 2.1 Basic definitions

We will define the Yang-Baxter equation and other relevant terms, following [2, 3, 4].

Let $R(u)$ be a parameter-dependent operator that acts in $\mathbb{C}^n \otimes \mathbb{C}^n$. The Yang-Baxter equation for the operator $R$ is

$$R^{k_1k_2}_{j_1j_2}(u)\,R^{j_1k_3}_{i_1j_3}(u+v)\,R^{j_2j_3}_{i_2i_3}(v) = R^{k_2k_3}_{j_2j_3}(v)\,R^{k_1j_3}_{j_1i_3}(u+v)\,R^{j_1j_2}_{i_1i_2}(u)\,, \tag{1}$$

where $u$ and $v$ are spectral parameters.

The operator $R$ is invariant with respect to simple Lie group $G$ (called $G$-invariant), if for each element $A$ of the Lie algebra associated with $G$ the following identity holds:

$$[A \otimes I + I \otimes A,\ R(u)] = 0. \tag{2}$$

The operator $R$ satisfies the unitary requirement, if such a function $\mu(u)$ exists that the following condition is met:

$$R^{k_1k_2}_{j_1j_2}(u)\,R^{j_2j_1}_{i_2i_1}(-u) = \mu(u)\,\delta^{k_1}_{i_1}\delta^{k_2}_{i_2}. \tag{3}$$

## 2.2 Important properties

It is easy to see that if an operator $R(u)$ is a solution of the Yang-Baxter equation (1), then $f(u)\,R(u)$ and $R(\alpha u)$ are also solutions for any arbitrary function $f(u)$ and parameter $\alpha$. In other words, the Yang-Baxter equation (1) allows argument scaling

$$u \to \alpha u, \tag{4}$$

and solution scaling

$$R(u) \to f(u)\,R(u)\,. \tag{5}$$

Let $I$ be the identity operator and let $P$ be the permutation operator, defined as follows:

$$I^{k_1k_2}_{i_1i_2} = \delta^{k_1}_{i_1}\delta^{k_2}_{i_2}, \quad P^{k_1k_2}_{i_1i_2} = \delta^{k_1}_{i_2}\delta^{k_2}_{i_1} \tag{6}$$

One can prove via a direct substitution that these two operators are constant solutions of (1), (2) and (3) .

## 2.3 Example

The solution of the Yang-Baxter equation (1) with $GL(n, \mathbb{C})$ symmetry is given by

$$R(u) = u \cdot I - P, \tag{7}$$

where $I$ and $P$ were defined previously in (6). Let us check that (7) is a solution of the Yang-Baxter equation (1):

$$\begin{aligned} LHS = R^{k_1k_2}_{j_1j_2}(u)\, R^{j_1k_3}_{i_1j_3}(u+v)\, R^{j_2j_3}_{i_2i_3}(v) = (u\cdot I - P)^{k_1k_2}_{j_1j_2}\left((u+v)\cdot I - P\right)^{j_1k_3}_{i_1j_3}(v\cdot I - P)^{j_2j_3}_{i_2i_3} = \\ = u(u+v)\,v\delta^{k_1}_{i_1}\delta^{k_2}_{i_2}\delta^{k_3}_{i_3} - u(u+v)\,\delta^{k_1}_{i_1}\delta^{k_2}_{i_3}\delta^{k_3}_{i_2} - uv\delta^{k_1}_{i_3}\delta^{k_2}_{i_2}\delta^{k_3}_{i_1} - (u+v)\,v\delta^{k_1}_{i_2}\delta^{k_2}_{i_1}\delta^{k_3}_{i_3} + \\ + u\delta^{k_1}_{i_2}\delta^{k_2}_{i_3}\delta^{k_3}_{i_1} + (u+v)\,\delta^{k_1}_{i_3}\delta^{k_2}_{i_1}\delta^{k_3}_{i_2} + v\delta^{k_1}_{i_2}\delta^{k_2}_{i_3}\delta^{k_3}_{i_1} - \delta^{k_1}_{i_3}\delta^{k_2}_{i_2}\delta^{k_3}_{i_1}, \end{aligned}$$

$$\begin{aligned} RHS = R^{k_2k_3}_{j_2j_3}(v)\, R^{k_1j_3}_{j_1i_3}(u+v)\, R^{j_1j_2}_{i_1i_2}(u) = (v\cdot I - P)^{k_2k_3}_{j_2j_3}\left((u+v)\cdot I - P\right)^{k_1j_3}_{j_1i_3}(u\cdot I - P)^{j_1j_2}_{i_1i_2} = \\ = u(u+v)\,v\delta^{k_1}_{i_1}\delta^{k_2}_{i_2}\delta^{k_3}_{i_3} - v(u+v)\,\delta^{k_1}_{i_2}\delta^{k_2}_{i_1}\delta^{k_3}_{i_3} - vu\delta^{k_1}_{i_3}\delta^{k_2}_{i_2}\delta^{k_3}_{i_1} - (u+v)\,u\delta^{k_1}_{i_1}\delta^{k_2}_{i_3}\delta^{k_3}_{i_2} + \\ + u\delta^{k_1}_{i_3}\delta^{k_2}_{i_1}\delta^{k_3}_{i_2} + (u+v)\,\delta^{k_1}_{i_2}\delta^{k_2}_{i_3}\delta^{k_3}_{i_1} + v\delta^{k_1}_{i_3}\delta^{k_2}_{i_1}\delta^{k_3}_{i_2} - \delta^{k_1}_{i_3}\delta^{k_2}_{i_2}\delta^{k_3}_{i_1}. \end{aligned}$$

The direct substitution yields that the right hand side and the left hand side are identical. Hence, (7) is indeed a solution of the Yang-Baxter equation (1). The $R$-matrix (7) is called Yang's $R$-matrix.

## 2.4 Other forms of the Yang-Baxter equation

One can rewrite the Yang-Baxter equation (1) in the form

$$R_{12}(u)\, R_{13}(u+v)\, R_{23}(v) = R_{23}(v)\, R_{13}(u+v)\, R_{12}(u),$$

where the indices denote the subspaces, in which $R$ acts nontrivially. With this notation, the unitary requirement (3) can expressed as follows:

$$R_{12}(u)\, R_{21}(-u) = \mu(u)\,\mathcal{I},$$

where $\mathcal{I}$ is the unit operator.

The braid form of the Yang-Baxter equation is

$$\check{R}_1\check{R}_2\check{R}_1 = \check{R}_2\check{R}_1\check{R}_2,$$

where $\check{R}_1 = P_{12}R_{12}$, $\check{R}_2 = P_{23}R_{23}$ is the braid form of the operator $R$, and $P_{12} = \delta^{k_1}_{i_2}\delta^{k_2}_{i_1}$, $P_{23} = \delta^{k_2}_{i_3}\delta^{k_3}_{i_2}$ are permutation operators that act on the corresponding subspaces.

# 3 Solution with $G_2$ symmetry

## 3.1 Exceptional Lie algebra $\mathfrak{g}_2$

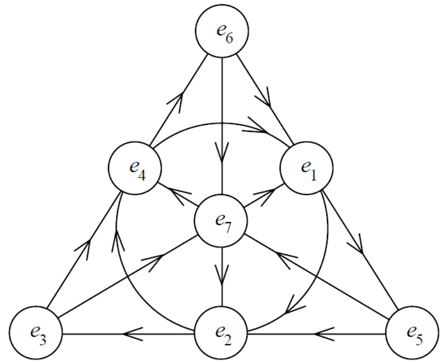
The exceptional Lie group $G_2$ can be realized as the automorphism group of the octonionic algebra $\mathbb{O}$ and its Lie algebra $\mathfrak{g}_2$ can be realized as derivations of $\mathbb{O}$.

We will refer to [5] for the notations of the octonions. Let $e_0 = 1, e_1, \ldots, e_7$ be the standard basis for $\mathbb{O}$ with the multiplication table, given in Tab. 1. The multiplication rules of the imaginary octonions can be depicted via the Fano plane (Fig. 1).

Table 1: Octonion Multiplication Table

| | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $-1$ | $e_4$ | $e_7$ | $-e_2$ | $e_6$ | $-e_5$ | $-e_3$ |
| $e_2$ | $-e_4$ | $-1$ | $e_5$ | $e_1$ | $-e_3$ | $e_7$ | $-e_6$ |
| $e_3$ | $-e_7$ | $-e_5$ | $-1$ | $e_6$ | $e_2$ | $-e_4$ | $e_1$ |
| $e_4$ | $e_2$ | $-e_1$ | $-e_6$ | $-1$ | $e_7$ | $e_3$ | $-e_5$ |
| $e_5$ | $-e_6$ | $e_3$ | $-e_2$ | $-e_7$ | $-1$ | $e_1$ | $e_4$ |
| $e_6$ | $e_5$ | $-e_7$ | $e_4$ | $-e_3$ | $-e_1$ | $-1$ | $e_2$ |
| $e_7$ | $e_3$ | $e_6$ | $-e_1$ | $e_5$ | $-e_4$ | $-e_2$ | $-1$ |

Figure 1: The Fano Plane




Let $f_{ijk}$ be the structure constants of octonions given by

$$e_i \cdot e_j = -\delta_{ij} + f_{ijk} e_k, \tag{8}$$

where indices $i,\ j,\ k \in \overline{1,\ 7}$.

From the multiplication table we can conclude that $f_{ijk}$ is antisymmetric in the first two indices. Using this fact, we obtain following identity:

$$e_i e_j + e_j e_i = -2\delta_{ij}. \tag{9}$$

The elements of the Lie algebra $\mathfrak{g}_2$ are derivations of $\mathbb{O}$, hence the following condition must be satisfied:

$$\forall d \in \mathfrak{g}_2\, \forall o_1,\ o_2 \in \mathbb{O} \quad d\left(o_1 \cdot o_2\right) = d\left(o_1\right) \cdot o_2 + o_1 \cdot d\left(o_2\right). \tag{10}$$

Let us denote the coordinates of $d\left(e_i\right)$ in the standard basis for $\mathbb{O}$ as $d_{ik}$:

$$d\left(e_i\right) = \sum_{k=0}^{7} d_{ik} e_k.$$

We calculate $d\left(1 \cdot 1\right)$ in two different ways, using (10):

$$\begin{aligned} &d\left(1 \cdot 1\right) = d\left(1\right), \\ &d\left(1 \cdot 1\right) = d\left(1\right) \cdot 1 + 1 \cdot d\left(1\right) = 2d\left(1\right). \end{aligned}$$

Consequently,

$$d\left(1\right) = 0. \tag{11}$$

In coordinate form

$$d_{0k} = 0\ \forall k \in \overline{0,\ 7}. \tag{12}$$

Now we calculate $d\left(e_i \cdot e_i\right)\ \forall i \in \overline{1,\ 7}$ in two ways, using (10):

$$\begin{aligned} &d\left(e_i \cdot e_i\right) = d\left(e_i^2\right) = d\left(-1\right) = -d\left(1\right) = 0, \\ &d\left(e_i \cdot e_i\right) = d\left(e_i\right) \cdot e_i + e_i \cdot d\left(e_i\right) = 2d_{i0} e_i + \sum_{k=1}^{7} d_{ik}\left(e_k e_i + e_i e_k\right) = 2d_{i0} e_i - 2d_{ii}. \end{aligned}$$

It follows that

$$d_{i0} = d_{ii} = 0. \tag{13}$$

We express $d\left(e_i\right)$, $i \in \overline{4,\ 7}$, via $d\left(e_j\right)$, $j \in \overline{1,\ 3}$, using (10):

$$\begin{aligned} d\left(e_4\right) = d\left(e_1 \cdot e_2\right) = d\left(e_1\right) e_2 + e_1 d\left(e_2\right), \\ d\left(e_5\right) = d\left(e_2 \cdot e_3\right) = d\left(e_2\right) e_3 + e_2 d\left(e_3\right), \\ d\left(e_6\right) = d\left(e_3 \cdot e_4\right) = d\left(e_3\right) e_4 + e_3 d\left(e_4\right), \\ d\left(e_7\right) = d\left(e_1 \cdot e_3\right) = d\left(e_1\right) e_3 + e_1 d\left(e_3\right). \end{aligned} \tag{14}$$

Let us substitute the coordinate form of $d(e_j)$, $j \in \overline{1,3}$, into (14) and use the relation (13). As a result, we obtain the following relations:

$$d_{12} + d_{21} = 0, \quad d_{23} + d_{32} = 0, \quad d_{15} - d_{27} + d_{34} = 0, \quad d_{13} + d_{31} = 0.$$

It follows that

$$d_{12} = -d_{21}, \quad d_{31} = -d_{13}, \quad d_{32} = -d_{23}, \quad d_{34} = d_{27} - d_{15}. \tag{15}$$

The relations (12), (13) and (14) are sufficient for the condition (10) to be satisfied. Thus, we have obtained the general form of a derivation of $\mathbb{O}$. We can now express the general form of an element $X$ of the Lie algebra $\mathfrak{g}_2$ as a matrix, which represents the action of derivation of the imaginary octonions $e_1, \ldots, e_7$. For the sake of convenience we rename the remaining independent components of $d(e_i)$:

$$\begin{array}{ccccccc} d_{12} \to p_1, & d_{13} \to p_2, & d_{14} \to p_3, & d_{15} \to p_4, & d_{16} \to p_5, & d_{17} \to p_6, & d_{23} \to p_7, \\ d_{24} \to p_8, & d_{25} \to p_9, & d_{26} \to p_{10}, & d_{27} \to p_{11}, & d_{35} \to p_{12}, & d_{36} \to p_{13}, & d_{37} \to p_{14}. \end{array} \tag{16}$$

After the replacement(16) we obtain the general form of $X \in \mathfrak{g}_2$:

$$X = \begin{pmatrix} 0 & -p_1 & -p_2 & -p_3 & -p_4 & -p_5 & -p_6 \\ p_1 & 0 & -p_7 & -p_8 & -p_9 & -p_{10} & -p_{11} \\ p_2 & p_7 & 0 & p_4 - p_{11} & -p_{12} & -p_{13} & -p_{14} \\ p_3 & p_8 & p_{11} - p_4 & 0 & p_2 + p_{10} & -p_6 - p_9 & p_5 - p_7 \\ p_4 & p_9 & p_{12} & -p_2 - p_{10} & 0 & p_8 + p_{14} & p_1 - p_{13} \\ p_5 & p_{10} & p_{13} & p_6 + p_9 & -p_8 - p_{14} & 0 & p_{12} - p_3 \\ p_6 & p_{11} & p_{14} & p_7 - p_5 & p_{13} - p_1 & p_3 - p_{12} & 0 \end{pmatrix}. \tag{17}$$

The basis for $\mathfrak{g}_2$ can be obtained by expressing the general form (17) as a sum of parameters $p_i$ multiplied by the basis matrices. However, in our case it is convenient to choose a basis, for which the Killing metric $g_{ab}$ is diagonal:

$$g_{ab} \propto \mathrm{Tr}\,(T_a T_b) = -\delta_{ab}. \tag{18}$$

The basis $\{T_a\}_{a=1}^{14}$ for $\mathfrak{g}_2$ was constructed with the help of the Wolfram Mathematica program presented in App. A.1. The basis for $\mathfrak{g}_2$ we obtained is presented in App. A.2.

### 3.2 Construction of the solution

Having obtained the basis of the Lie algebra $\mathfrak{g}_2$ in 3.1, we can now define the split Casimir operator as follows:

$$C_{i_1 i_2}^{k_1 k_2} \propto -g^{ab} \left(T_a\right)_{i_1}^{k_1} \left(T_b\right)_{i_2}^{k_2}, \tag{19}$$

where $g^{ab}$ is the inverse of the Killing metric (18). Tensor (19) satisfies (2) for all $A \in \mathfrak{g}_2$. To construct the solution of the Yang-Baxter equation we will use the tensors $I$ and $P$, which we defined in (6) and which satisfy (2) as well, and two more tensors – the symmetric and antisymmetric parts of tensor $C$:

$$S_{i_1 i_2}^{k_1 k_2} = C_{i_1 i_2}^{(k_1 k_2)} = \frac{1}{2}\left(C_{i_1 i_2}^{k_1 k_2} + C_{i_1 i_2}^{k_2 k_1}\right), \quad As_{i_1 i_2}^{k_1 k_2} = C_{i_1 i_2}^{[k_1 k_2]} = \frac{1}{2}\left(C_{i_1 i_2}^{k_1 k_2} - C_{i_1 i_2}^{k_2 k_1}\right). \tag{20}$$

Since $C_{i_1 i_2}^{k_1 k_2}$ is symmetric with respect to simultaneous transposition of upper and lower indices, tensors $S$ and $As$ are symmetric and antisymmetric both in upper and lower indices.

We will seek the solution of the Yang-Baxter equation (1) in the form

$$R_{i_1 i_2}^{k_1 k_2}(u) = f_1(u)\, I_{i_1 i_2}^{k_1 k_2} + f_2(u)\, P_{i_1 i_2}^{k_1 k_2} + f_3(u)\, S_{i_1 i_2}^{k_1 k_2} + f_4(u)\, As_{i_1 i_2}^{k_1 k_2}. \tag{21}$$

For the functions in (21) we will use the following substitution:

$$f_1(u) = u^3 + a_{12}u^2 + a_{11}u + a_{10}, \quad f_2(u) = a_{21}u + a_{20}, \quad f_3(u) = a_{32}u^2 + a_{31}u + a_{30}, \quad f_4(u) = a_{42}u^2 + a_{41}u + a_{40}. \tag{22}$$

Plugging (21) into the Yang-Baxter equation (1), we obtain a system of equations for the unknown coefficients of functions in (22). We used the Wolfram Mathematica program presented in App. A.3, to find the coefficients:

$$a_{12} = -\frac{3}{2}a_{42},\ a_{11} = \frac{5}{9}a_{42}^2,\ a_{10} = 0,\ a_{21} = \frac{1}{6}a_{42}^2,\ a_{20} = -\frac{1}{9}a_{42}^3,\ a_{32} = a_{42},\ a_{31} = -\frac{2}{3}a_{42}^2,\ a_{30} = 0,\ a_{41} = -a_{42}^2,\ a_{40} = 0. \tag{23}$$

Thus, the solution of the Yang-Baxter equation is

$$R(u) = \left(u^3 - \frac{3}{2}a_{42}u^2 + \frac{5}{9}a_{42}^2 u\right) I + \left(\frac{1}{6}a_{42}^2 u - \frac{1}{9}a_{42}^3\right) P + \left(a_{42}u^2 - \frac{2}{3}a_{42}^2 u\right) S + \left(a_{42}u^2 - a_{42}^2 u\right) As. \tag{24}$$

The coefficient $a_{42}$ can be chosen arbitrarily, as follows from the argument renormalization (4). For the sake of convenience we set $a_{42} = 6$ and write the corresponding solution as

$$R(u) = \left(u^3 - 9u^2 + 20u\right) I + (6u - 24) P + \left(6u^2 - 24u\right) S + \left(6u^2 - 36u\right) As. \tag{25}$$

The solution (25) of the Yang-Baxter equation satisfies the unitary requirement (3), where

$$\mu(u) = -(u-6)(u-4)(u-1)(u+1)(u+4)(u+6).$$

### 3.3 Analysis of representations

In case of $G_2$, the operator $R$ acts on the space $\mathbb{C}^7 \otimes \mathbb{C}^7$ of dimension 49. This space can be decomposed into a direct sum of spaces of symmetric and antisymmetric tensors, we will denote them as [**28**] and [**21**], according to their dimensions. The $G_2$ group has 4 irreps of dimension less than 49, we will denote them in a similar fashion: [**1**], [**7**], [**14**] and [**27**]. Now we decompose the spaces [**28**] and [**21**] into a direct sum of spaces of irreps of $G_2$. It is easy to guess the decomposition from the dimensions of the spaces: [**28**] = [**1**] + [**27**], [**21**] = [**7**] + [**14**].

It follows from this decomposition that the operator $R$ should be constructed from the projectors onto these irreps, which we will denote $\mathcal{P}_1$, $\mathcal{P}_7$, $\mathcal{P}_{14}$ and $\mathcal{P}_{27}$. The projectors satisfy the following relations:

$$\mathcal{P}_i \mathcal{P}_j = \delta_{ij} \mathcal{P}_i, \tag{26a}$$

$$\sum_i \mathcal{P}_i = \mathcal{I}, \tag{26b}$$

where $i,\, j \in \{1,\, 7,\, 14,\, 28\}$ and $\mathcal{I}$ is the unit operator. From these relations one can work out the formulas for the projectors:

$$\begin{aligned} \mathcal{P}_1 &= \frac{1}{14} \cdot (I + P) + \frac{3}{7} \cdot S, \\ \mathcal{P}_7 &= As, \\ \mathcal{P}_{14} &= \frac{1}{2}(I - P) - As, \\ \mathcal{P}_{27} &= \frac{3}{7}(I + P) - \frac{3}{7} \cdot S. \end{aligned} \tag{27}$$

### 3.4 Tensor relations and expressions for projectors

Let us consider a symmetric tensor $K$:

$$K_{i_1 i_2}^{k_1 k_2} = \delta^{k_1 k_2} \delta_{i_1 i_2}. \tag{28}$$

It turns out that $K$ can be represented as a linear combination of symmetric tensors $I + P$ and $S$:

$$K = \frac{1}{2} \cdot (I + P) + 3 \cdot S. \tag{29}$$

From (29) and (27) we can work out the formulas for the projectors, involving tensor $K$:

$$\begin{aligned} \mathcal{P}_1 &= \frac{1}{7} K, \\ \mathcal{P}_{27} &= \frac{1}{2}(I + P) - \frac{1}{7} K. \end{aligned} \tag{30}$$

Let us consider tensor $F$:

$$F_{i_1 i_2}^{k_1 k_2} = f^{k_1 k_2 m} f_{i_1 i_2 m},$$

where $f_{ijk}$ are the octonionic structure coefficients, defined in (8). Tensor $F$ is antisymmetric and turns out to be proportional to $As$:

$$F = 6 \cdot As. \tag{31}$$

From (31) and (27) we can find the relations between the projectors and tensor $F$:

$$\begin{aligned}\mathcal{P}_7 &= \frac{1}{6}F, \\ \mathcal{P}_{14} &= \frac{1}{2}(I-P) - \frac{1}{6}F.\end{aligned} \tag{32}$$

In fact, irrep [**7**] is action of $G_2$ on the imaginary octonions, so $\mathcal{P}_7$ is proportional to $F$.

Let us consider the modified Casimir tensor (compare to (19)):

$$\tilde{C}^{k_1 k_2}_{i_1 i_2} = (T_a)^{k_1 k_2}(T_a)_{i_1 i_2}. \tag{33}$$

Tensor $\tilde{C}$ is antisymmetric and can be expressed as a linear combination of antisymmetric tensors $I-P$ and $As$:

$$\tilde{C} = \frac{1}{2}(I-P) - As. \tag{34}$$

From (34) and (27) we can see that

$$\mathcal{P}_{14} = \tilde{C}.$$

Indeed, from the definition (33) of the modified Casimir tensor we can see that $\tilde{C}$ is the projector onto the adjoint representation, which we denoted [**14**].

The expressions for the projectors (30) and (32) are the same as in [2]. Using the relations (27), one can show that our result (25) and that, obtained in [2], are equivalent up to scaling.

# 4 Solution with $F_4$ symmetry

## 4.1 Exceptional Lie algebra $\mathfrak{f}_4$

We construct the $\mathfrak{f}_4$ algebra, following [6].

The exceptional Lie group $F_4$ is the automorphism group of the exceptional Jordan Algebra $J_3$, which consists of $3\times 3$ octonionic hermitian matrices. The general form of an element $A \in J_3$ is

$$A = \begin{pmatrix} a_1 & o_1 & o_2 \\ \overline{o_1} & a_2 & o_3 \\ \overline{o_2} & \overline{o_3} & a_3 \end{pmatrix}, \tag{35}$$

where $a_i \in \mathbb{R}$, $o_i \in \mathbb{O}$. The product in $J_3$ is defined by the following formula:

$$A \circ B = \frac{1}{2}(A\cdot B + B\cdot A) \ \forall A,\, B \in J_3. \tag{36}$$

Let us validate the definition (36) by calculating the product of two matrices $A,\, B \in J_3$ of general form:

$$A = \sqrt{2}\begin{pmatrix} a_1 & o_1 & o_2 \\ \overline{o_1} & a_2 & o_3 \\ \overline{o_2} & \overline{o_3} & a_3 \end{pmatrix}, \quad B = \sqrt{2}\begin{pmatrix} b_1 & p_1 & p_2 \\ \overline{p_1} & b_2 & p_3 \\ \overline{p_2} & \overline{p_3} & b_3 \end{pmatrix}.$$

The formula (36) yields

$$\begin{pmatrix} 2a_1b_1 + o_1\overline{p_1} + o_2\overline{p_2} + p_1\overline{o_1} + p_2\overline{o_2} & (a_1+a_2)\,p_1 + (b_1+b_2)\,o_1 + o_2\overline{p_3} + p_2\overline{o_3} & (a_1+a_3)\,p_2 + (b_1+b_3)\,o_2 + o_1p_3 + p_1o_3 \\ (a_1+a_2)\,\overline{p_1} + (b_1+b_2)\,\overline{o_1} + o_3\overline{p_2} + p_3\overline{o_2} & 2a_2b_2 + \overline{o_1}p_1 + o_3\overline{p_3} + \overline{p_1}o_1 + p_3\overline{o_3} & (a_2+a_3)\,p_3 + (b_2+b_3)\,o_3 + \overline{o_1}p_2 + o_3b_3 \\ (a_1+a_3)\,\overline{p_2} + (b_1+b_3)\,\overline{o_2} + \overline{o_3}\,\overline{p_1} + \overline{p_3}\,\overline{o_1} & (a_2+a_3)\,\overline{p_3} + (b_2+b_3)\,\overline{o_3} + \overline{o_2}p_1 + \overline{p_2}o_1 & 2a_3b_3 + \overline{o_2}p_2 + \overline{o_3}p_3 + \overline{p_2}o_2 + \overline{p_3}o_3 \end{pmatrix},$$

which is an element of $J_3$, since for two arbitrary octonions the following identity holds:

$$\overline{o_1 o_2} = \overline{o_2}\,\overline{o_1} \ \forall o_1,\, o_2 \in \mathbb{O}.$$

Therefore, the formula (36) gives a valid definition of multiplication in $J_3$.

The exceptional Lie algebra $\mathfrak{f}_4$ is the algebra of derivations of $J_3$. Note, that any matrix in $J_3$ is defined by 27 real values: 3 real values on the main diagonal and 8 components of the 3 octonions above the main diagonal. This one-to-one correspondence allows us to treat $\mathbb{R}^{27}$ vectors as $J_3$ matrices and define product in $\mathbb{R}^{27}$ in the following fashion:

$$x \circ y = \Phi\left(\Phi^{-1}(x) \circ \Phi^{-1}(y)\right) \ \forall x,\, y \in \mathbb{R}^{27},$$

where $\Phi:\ J_3 \to \mathbb{R}^{27}$ and $\Phi^{-1}:\ \mathbb{R}^{27} \to J_3$ denote the aforementioned identification. Using these conventions,$\mathfrak{f}_4$ can be represented as an algebra of real $27\times 27$ matrices. Each element $M \in \mathfrak{f}_4$ is a derivation of $J_3$, hence the identity must hold:

$$M\cdot(x\circ y) = (M\cdot x)\circ y + x\circ(M\cdot y)\ \ \forall x,\, y\in\mathbb{R}^{27}.$$

The relations between the components of the matrix $M$ that follow from this identity can be solved to obtain the general form of an element $M \in \mathfrak{f}_4$.

We construct a basis $\{T_a\}_{a=1}^{52}$ in $\mathfrak{f}_4$ that satisfies two conditions: the last (27th) row and column of all basis matrices are zero and the Killing metric $g_{ab}$ is proportional to

$$\mathrm{Tr}\,(T_a T_b) = -6\delta_{ab}.$$

This basis was constructed by means of Wolfram Mathematica, the program is presented in App. B.1. The basis for $\mathfrak{f}_4$ we obtained is presented in App. B.2.

## 4.2 Construction of the solution

Following the construction of the solution with $G_2$ symmetry, we still use symmetric $S$ and antisymmetric $As$ parts of the Casimir tensor $C$ (see (19) and (20)):

$$C_{i_1 i_2}^{k_1 k_2} \propto g^{ab}\,(T_a)_{i_1}^{k_1}\,(T_b)_{i_2}^{k_2}\,,\quad S_{i_1 i_2}^{k_1 k_2} = \frac{1}{2}\left(C_{i_1 i_2}^{k_1 k_2} + C_{i_1 i_2}^{k_2 k_1}\right),\quad As_{i_1 i_2}^{k_1 k_2} = \frac{1}{2}\left(C_{i_1 i_2}^{k_1 k_2} - C_{i_1 i_2}^{k_2 k_1}\right),$$

where $\{T_a\}$ is the $\mathfrak{f}_4$ basis, which was obtained in the previous section.

We will also use tensor $K$ as defined in (28):

$$K_{i_1 i_2}^{k_1 k_2} = \delta^{k_1 k_2}\delta_{i_1 i_2}.$$

We will seek the solution of the Yang-Baxter equation(1) in the form

$$R_{i_1 i_2}^{k_1 k_2}(u) = f_1(u)\, I_{i_1 i_2}^{k_1 k_2} + f_2(u)\, P_{i_1 i_2}^{k_1 k_2} + f_3(u)\, K_{i_1 i_2}^{k_1 k_2} + f_4(u)\, S_{i_1 i_2}^{k_1 k_2} + f_5(u)\, As_{i_1 i_2}^{k_1 k_2}. \tag{37}$$

For the functions in (37) we will use the following substitution:

$$f_1(u) = -u + \frac{a_1 u}{u-p_1},\quad f_2(u) = 1 + \frac{a_2 u}{u-p_1},\quad f_3(u) = \frac{c_1 u}{u-p_1} + \frac{c_2 u}{u-p_2} + \frac{c_3 u}{u-p_3},\quad f_4(u) = \frac{d_1 u}{u-p_1},\quad f_5(u) = \frac{d_2 u}{u-p_2}. \tag{38}$$

Plugging (37) into the Yang-Baxter equation (1), we obtain a system of equations for the unknown coefficients of functions in (38). We used the Wolfram Mathematica program presented in App. B.3, to find the coefficients:

$$a_1 = -1,\quad a_2 = -1,\quad c_1 = 1,\quad c_2 = \frac{3}{5},\quad c_3 = -\frac{8}{5},\quad d_1 = -2,\quad d_2 = -2,\quad p_1 = 6,\quad p_2 = 4,\quad p_3 = 9. \tag{39}$$

Thus, the solution of the Yang-Baxter equation is

$$R(u) = \left(-u - \frac{u}{u-6}\right)\cdot I + \left(1 - \frac{u}{u-6}\right)\cdot P + \left(\frac{u}{u-6} + \frac{3u}{5(u-4)} - \frac{8u}{5(u-9)}\right)\cdot K - \frac{2u}{u-6}\cdot S - \frac{2u}{u-4}\cdot As. \tag{40}$$

The unitary requirement (3) for (40) is satisfied when

$$\mu(u) = 1 - u^2.$$

## 4.3 Analysis of representations

In case of $F_4$, the operator $R$ acts on the space $\mathbb{C}^{26}\otimes\mathbb{C}^{26}$ of dimension 676. This space can be decomposed into a direct sum of spaces of symmetric and antisymmetric tensors, we will denote them as [**351**] and [**325**], according to their dimensions. The $F_4$ group has 5 irreps of dimension less than 676, we will denote them in a similar fashion: [**1**], [**26**], [**52**], [**273**] and [**324**]. Now let us decompose the spaces [**351**] and [**325**] into a direct sum of spaces of irreps of $F_4$. The symmetric space [**351**] includes the trivial irrep [**1**]. It is easy to guess the decomposition from the dimensions of the spaces: [**351**] = [**1**] + [**26**] + [**324**], [**325**] = [**52**] + [**273**].

It follows from this decomposition that the operator $R$ should be constructed from the projectors onto these irreps, which we will denote $\mathcal{P}_1$, $\mathcal{P}_{26}$, $\mathcal{P}_{52}$, $\mathcal{P}_{273}$ and $\mathcal{P}_{324}$. The projectors satisfy the following relations:

$$\mathcal{P}_i\mathcal{P}_j = \delta_{ij}\mathcal{P}_i,$$
$$\sum_i \mathcal{P}_i = \mathcal{I},$$

where $i,\, j\in\{1,\, 26,\, 52,\, 273,\, 324\}$ and $\mathcal{I}$ is the unit operator.

### 4.4 Tensor relations

Let us consider the modified Casimir tensor, defined as in (33):

$$\tilde{C}^{k_1 k_2}_{i_1 i_2} = (T_a)^{k_1 k_2} (T_a)_{i_1 i_2} \,.$$

Tensor $\tilde{C}$ is antisymmetric and turns out to be proportional to $As$:

$$\tilde{C} = 2 \cdot As. \tag{41}$$

Since $\tilde{C}$ is the projector on the adjoint representation, as noted in (3.4), we obtain

$$\mathcal{P}_{52} = \tilde{C} = 2 \cdot As. \tag{42}$$

We can see a direct analogy between $\mathfrak{g}_2$ and $\mathfrak{f}_4$, which we realized as derivations of $\mathbb{O}$ and $J_3$ respectively. In case of $\mathfrak{g}_2$ we defined tensor $F$ using the octonionic structure constants $f_{ijk}$. Using an analogous formula, we define tensor $D$ for $\mathfrak{f}_4$:

$$D^{k_1 k_2}_{i_1 i_2} = d^{k_1 k_2 m} d_{i_1 i_2 m}, \tag{43}$$

where $d_{abc}$ are structure coefficients of $J_3$ algebra. We calculate $d_{abc}$ in basis $\{j_a\}_{a=1}^{26}$ that corresponds to the construction of $\mathfrak{f}_4$ we presented in 4.1:

$$j_a \circ j_b = c\delta_{ab} + d_{abc} j_c, \tag{44}$$

where $\circ$ denites the product (36). Since the product in $J_3$ is commutative, the structure constants are symmetric in the first two indices: $d_{abc} = d_{bac}$. Thus, $D$ is symmetric. As it turns out, tensor $D$ can be represented as a linear combination of tensors that we used to find the solution:

$$D = \frac{1}{3} \cdot (I + P) - \frac{1}{3} \cdot K + \frac{2}{3} \cdot S. \tag{45}$$

## 5 Solution with $E_6$ symmetry

Up to now the explicit form of the solution of the Yang-Baxter equation (1) with the $E_6$ symmetry remains unknown. We will provide some results and considerations regarding this problem.

### 5.1 Exceptional Lie algebra $\mathfrak{e}_6$

We define $\mathfrak{e}_6$ and construct the basis, following [7].

The exceptional Lie algebra $\mathfrak{e}_6$ is spanned by the algebra of derivations of $J_3$ and the algebra $\mathfrak{R}_Y$ of right multiplications of traceless $J_3$ matrices. Hence, the basis for $\mathfrak{R}_Y$ complements the basis for $\mathfrak{f}_4$, which we obtained in section 4.1, to the basis for $\mathfrak{e}_6$. For convenience, we

$$\mathrm{Tr}\,(T_a T_b) = -6\delta_{ab}.$$

The complete basis for $\mathfrak{e}_6$ was obtained via Wolfram Mathematica, the program code is presented in App. C.1.

### 5.2 Analysis of representations

In case of $E_6$, the operator $R$ acts on the space $\mathbb{C}^{27} \otimes \mathbb{C}^{27}$ of dimension 729. This space can be decomposed into a direct sum of spaces of symmetric and antisymmetric tensors, we will denote them as $[\mathbf{378}]$ and $[\mathbf{351}]$, according to their dimensions. The $E_6$ group has 9 irreps of dimension less or equal to 729, we will denote them in a similar fashion: $[\mathbf{1}]$, $[\mathbf{27}]$ and $[\overline{\mathbf{27}}]$ (2 reps), $[\mathbf{78}]$, $[\mathbf{351}]_{a,b,c,d}$ (4 reps) and $[\mathbf{650}]$. Now let us decompose the spaces $[\mathbf{378}]$ and $[\mathbf{351}]$ into a direct sum of irreps of $E_6$, using each irrep only once. It is easy to guess the decomposition from the dimensions of the spaces: $[\mathbf{378}] = [\mathbf{351}]_a + [\mathbf{27}]$, $[\mathbf{351}] = [\mathbf{351}]_b$.

It follows from this decomposition that the operator $R$ should be constructed from the projectors onto these irreps, which we will denote $\mathcal{P}_{27}$, $\mathcal{P}_{351a}$ and $\mathcal{P}_{351b}$. The projectors satisfy the following relations:

$$\mathcal{P}_i \mathcal{P}_j = \delta_{ij} \mathcal{P}_i,$$
$$\sum_i \mathcal{P}_i = \mathcal{I},$$

where $i,\, j \in \{27,\, 351a,\, 351b\}$ and $\mathcal{I}$ is the unit operator.

### 5.3 Tensor relations

We know two invariant tensors $I$ and $P$. Thus, at least one more tensor is required in order to construct the projectors, which is probably the Casimir operator. Knowing all the tensors that constitute the projectors $\mathcal{P}_{27}$, $\mathcal{P}_{351a}$ and $\mathcal{P}_{351b}$, it is not hard to find the solution of the Yang-Baxter equation. In fact, the structure of the solution with $E_6$ symmetry in terms of projectors is given in [3].

## 6 Conclusion

In this work we have found the solutions of Yang-Baxter equation with $G_2$ and $F_4$ symmetry, given in (25) and (40). The structure of these solutions was analyzed from the point of view of representation theory. We have also provided some thoughts based on representation theory about the form of the solution with $E_6$ symmetry, which remains to be found in the future.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# A Solution with $G_2$ symmetry

## A.1 Construction of a basis for $\mathfrak{g}_2$

```
octBaseNames=Array[e,8];
octBase=IdentityMatrix[8];
octListForm[x_]:=Coefficient[x,octBaseNames];
octMultiplicationTable={
        {e[1],e[2],e[3],e[4],e[5],e[6],e[7],e[8]},
        {e[2],-e[1],e[5],e[8],-e[3],e[7],-e[6],-e[4]},
        {e[3],-e[5],-e[1],e[6],e[2],-e[4],e[8],-e[7]},
        {e[4],-e[8],-e[6],-e[1],e[7],e[3],-e[5],e[2]},
        {e[5],e[3],-e[2],-e[7],-e[1],e[8],e[4],-e[6]},
        {e[6],-e[7],e[4],-e[3],-e[8],-e[1],e[2],e[5]},
        {e[7],e[6],-e[8],e[5],-e[4],-e[2],-e[1],e[3]},
        {e[8],e[4],e[7],-e[2],e[6],-e[5],-e[3],-e[1]}};
octProduct[x_,y_]:=octListForm[Sum[x[[i]]*y[[j]]*octMultiplicationTable[[i,j]],{i,8},{j,8}]];
octConjugate[x_]:=Join[{x[[1]]},-x[[2;;8]]];
octNormSquare[x_]:=Sum[x[[i]]^2,{i,8}];

diffMatrTransposed={
        ConstantArray[0,8],
        Join[{0,0},Array[a,6,3]],
        Join[{0,b[2],0},Array[b,5,4]],
        Join[{0,c[2],c[3],0},Array[c,4,5]],
        {},{},{},{}};
diffBaseOct[i_]:=diffMatrTransposed[[i]];
diffProdBaseOcts[i_,j_]:=octProduct[diffBaseOct[i],octBase[[j]]]+
        octProduct[octBase[[i]],diffBaseOct[j]];
recalcDiffMatrT[]:=Module[{},
        diffMatrTransposed[[5]]=diffProdBaseOcts[2,3];
        diffMatrTransposed[[6]]=diffProdBaseOcts[3,4];
        diffMatrTransposed[[7]]=diffProdBaseOcts[4,5];
        diffMatrTransposed[[8]]=diffProdBaseOcts[2,4];];
recalcDiffMatrT[];

paramExclude={b[2],c[3],c[5],c[2]};
paramSubstitution=Solve[Table[diffMatrTransposed[[i,1]]==0,{i,5,8}],paramExclude][[1]];
diffMatrTransposed=diffMatrTransposed/.paramSubstitution;
recalcDiffMatrT[];

ruleRenameParams=Thread[Join[Array[a,6,3],Array[b,5,4],Array[c,3,6]]->Array[p,14]];
g2GeneralForm=Drop[Transpose[diffMatrTransposed]/.ruleRenameParams,{1},{1}];

rulesReplaceBase=Table[{p[i]->1,p[x_]->0},{i,1,14}];
g2StandardBase=g2GeneralForm/.rulesReplaceBase;
        g2OrthonormalBase=Join[g2StandardBase[[1;;6]]/2,
        {g2StandardBase[[7]]+g2StandardBase[[5]]/2}/Sqrt[3],
        {g2StandardBase[[8]]/2},
        {g2StandardBase[[9]]-g2StandardBase[[6]]/2,
        g2StandardBase[[10]]-g2StandardBase[[2]]/2,
        g2StandardBase[[11]]+g2StandardBase[[4]]/2,
        g2StandardBase[[12]]+g2StandardBase[[3]]/2,
        g2StandardBase[[13]]+g2StandardBase[[1]]/2,
        g2StandardBase[[14]]-g2StandardBase[[8]]/2}/Sqrt[3]];
```

### A.2 Basis for $\mathfrak{g}_2$

The basis for $\mathfrak{g}_2$ we obtained is

$$\begin{aligned}
T_1 &= \frac{1}{2}M_{57} - \frac{1}{2}M_{12}, & T_2 &= \frac{1}{2}M_{45} - \frac{1}{2}M_{13},\\
T_3 &= -\frac{1}{2}M_{14} - \frac{1}{2}M_{67}, & T_4 &= \frac{1}{2}M_{34} - \frac{1}{2}M_{15},\\
T_5 &= \frac{1}{2}M_{47} - \frac{1}{2}M_{16}, & T_6 &= -\frac{1}{2}M_{17} - \frac{1}{2}M_{46},\\
T_7 &= -\frac{M_{16}}{2\sqrt{3}} - \frac{M_{23}}{\sqrt{3}} - \frac{M_{47}}{2\sqrt{3}}, & T_8 &= \frac{1}{2}M_{56} - \frac{1}{2}M_{24},\\
T_9 &= \frac{M_{17}}{2\sqrt{3}} - \frac{M_{25}}{\sqrt{3}} - \frac{M_{46}}{2\sqrt{3}}, & T_{10} &= \frac{M_{13}}{2\sqrt{3}} - \frac{M_{26}}{\sqrt{3}} + \frac{M_{45}}{2\sqrt{3}},\\
T_{11} &= -\frac{M_{15}}{2\sqrt{3}} - \frac{M_{27}}{\sqrt{3}} - \frac{M_{34}}{2\sqrt{3}}, & T_{12} &= -\frac{M_{14}}{2\sqrt{3}} - \frac{M_{35}}{\sqrt{3}} + \frac{M_{67}}{2\sqrt{3}},\\
T_{13} &= -\frac{M_{12}}{2\sqrt{3}} - \frac{M_{36}}{\sqrt{3}} - \frac{M_{57}}{2\sqrt{3}}, & T_{14} &= \frac{M_{24}}{2\sqrt{3}} - \frac{M_{37}}{\sqrt{3}} + \frac{M_{56}}{2\sqrt{3}},
\end{aligned}$$

where $M_{ij} = E_{ij} - E_{ji}$ are generators of $\mathfrak{so}(7)$.

### A.3 Solution of the Yang-Baxter equation

```
tensI=Table[KroneckerDelta[k1,i1]*KroneckerDelta[k2,i2],{k1,7},{k2,7},{i1,7},{i2,7}];
tensP=Table[KroneckerDelta[k1,i2]*KroneckerDelta[k2,i1],{k1,7},{k2,7},{i1,7},{i2,7}];
tensC=Table[Sum[g2OrthonormalBase[[a,k1,i1]]*g2OrthonormalBase[[a,k2,i2]],{a,14}],
        {k1,7},{k2,7},{i1,7},{i2,7}];
tensS=Table[(tensC[[k1,k2,i1,i2]]+tensC[[k2,k1,i1,i2]])/2,{k1,7},{k2,7},{i1,7},{i2,7}];
tensAs=Table[(tensC[[k1,k2,i1,i2]]-tensC[[k2,k1,i1,i2]])/2,{k1,7},{k2,7},{i1,7},{i2,7}];

params={a12,a11,a10,a21,a20,a32,a31,a30,a42,a41,a40};
tensR[x_][k1_,k2_,i1_,i2_]:=tensR[x][k1,k2,i1,i2]=
        tensI[[k1,k2,i1,i2]]*(x^3+a12*x^2+a11*x+a10)+
        tensP[[k1,k2,i1,i2]]*(a21*x+a20)+
        tensS[[k1,k2,i1,i2]]*(a32*x^2+a31*x+a30)+
        tensAs[[k1,k2,i1,i2]]*(a42*x^2+a41*x+a40);

eqnYB[k1_,k2_,k3_,i1_,i2_,i3_]:=
        Sum[tensR[u][k1,k2,j1,j2]*tensR[u+v][j1,k3,i1,j3]*tensR[v][j2,j3,i2,i3]-
        tensR[v][k2,k3,j2,j3]*tensR[u+v][k1,j3,j1,i3]*tensR[u][j1,j2,i1,i2],{j1,7},{j2,7},{j3,7}];

coeffs={};
Table[coeffs=Union[coeffs,
        Replace[CoefficientRules[eqnYB[k1,k2,k3,i1,i2,i3],{u,v}],{Rule[List[u_,v_],c_]->c},2]];,
        {k1,7},{k2,7},{k3,3},{i1,1},{i2,1},{i3,1}];
solns=Solve[Table[coeffs[[i]]==0,{i,Length[coeffs]}],params];
soln=Select[solns,!MemberQ[#,a32->0]&][[1]];
```

## B Solution with $F_4$ symmetry

### B.1 Construction of a basis for $\mathfrak{f}_4$

```
octBaseNames=Array[e,8];
octBase=IdentityMatrix[8];
octListForm[x_]:=Coefficient[x,octBaseNames];
octMultiplicationTable={
        {e[1],e[2],e[3],e[4],e[5],e[6],e[7],e[8]},
        {e[2],-e[1],e[5],e[8],-e[3],e[7],-e[6],-e[4]},
        {e[3],-e[5],-e[1],e[6],e[2],-e[4],e[8],-e[7]},
```

```
        {e[4],-e[8],-e[6],-e[1],e[7],e[3],-e[5],e[2]},
        {e[5],e[3],-e[2],-e[7],-e[1],e[8],e[4],-e[6]},
        {e[6],-e[7],e[4],-e[3],-e[8],-e[1],e[2],e[5]},
        {e[7],e[6],-e[8],e[5],-e[4],-e[2],-e[1],e[3]},
        {e[8],e[4],e[7],-e[2],e[6],-e[5],-e[3],-e[1]}};
octProduct[x_,y_]:=octListForm[Sum[x[[i]]*y[[j]]*octMultiplicationTable[[i,j]],{i,8},{j,8}]];
octConjugate[x_]:=Join[{x[[1]]},-x[[2;;8]]];
octNormSquare[x_]:=Sum[x[[i]]^2,{i,8}];

jordanToR27[AA_]:=Join[
        {AA[[1,1,1]]},AA[[1,2,1;;8]],AA[[1,3,1;;8]],{AA[[2,2,1]]},AA[[2,3,1;;8]],{AA[[3,3,1]]}];
r27ToJordan[vv_]:=Module[{a1,a2,a3,o1,o2,o3,o1c,o2c,o3c},
        a1=a2=a3=ConstantArray[0,8];
        a1[[1]]=vv[[1]];a2[[1]]=vv[[18]];a3[[1]]=vv[[27]];
        o1=vv[[2;;9]];o2=vv[[10;;17]];o3=vv[[19;;26]];
        o1c=octConjugate[o1];o2c=octConjugate[o2];o3c=octConjugate[o3];
        {{a1,o1,o2},{o1c,a2,o3},{o2c,o3c,a3}}];
octMatrProduct[a_,b_]:=Table[Sum[octProduct[a[[i,k]],b[[k,j]]],{k,3}],{i,3},{j,3}];
jordanProduct[a_,b_]:=1/2(octMatrProduct[a,b]+octMatrProduct[b,a]);
r27Product[x_,y_]:=jordanToR27[jordanProduct[r27ToJordan[x],r27ToJordan[y]]];

X=Array[x,27];
Y=Array[y,27];
M=Array[m,{27,27}];
diff=M.r27Product[X,Y]-r27Product[M.X,Y]-r27Product[X,M.Y];
elemsDiff=Flatten[Table[x[i]*y[j],{i,27},{j,27}]];
coeffsDiff=DeleteCases[Flatten[Table[Coefficient[diff[[k]],elemsDiff],{k,27}]],0];
solM=Solve[Table[coeffsDiff[[i]]==0,{i,Length[coeffsDiff]}],Flatten[M]];
M=M/.solM[[1]];
freeParams=Union[Cases[M,m[x_,y_],Infinity]];
ruleRenameParams=Thread[freeParams->Array[p,Length[freeParams]]];
f4GeneralForm=M/.ruleRenameParams;

rulesReplaceBase=Table[p[j]->KroneckerDelta[i,j],{i,Length[freeParams]},{j,Length[freeParams]}];
f4StandardBase=f4GeneralForm/.rulesReplaceBase;

killingForm[a_,b_]:=Sum[a[[i,j]]*b[[j,i]],{i,27},{j,27}];
killingNormSquare[a_]:=killingForm[a,a];
killingProject[a_,b_]:=killingForm[a,b]/killingForm[b,b]*b;
f4OrthogonalBase={};
Table[AppendTo[f4OrthogonalBase,f4StandardBase[[i]]-
        Sum[killingProject[f4StandardBase[[i]],f4OrthogonalBase[[j]]],{j,i-1}]];,
        {i,Length[f4StandardBase]}];

f4OrthonormalBase=Table[
        Sqrt[-6/killingNormSquare[f4OrthogonalBase[[i]]]]*f4OrthogonalBase[[i]],{i,52}];
rot1Mat27=EE=IdentityMatrix[27];
rot1Mat27[[1]]=(EE[[1]]-EE[[18]])/Sqrt[2];
rot1Mat27[[18]]=(EE[[1]]+EE[[18]]-2*EE[[27]])/Sqrt[6];
rot1Mat27[[27]]=(EE[[1]]+EE[[18]]+EE[[27]])/Sqrt[3];
invRot1Mat27=Transpose[rot1Mat27];
f4OrthonormalBase=Table[Drop[rot1Mat27.f4OrthonormalBase[[i]].invRot1Mat27,{27},{27}],{i,52}];
rot2Mat26=IdentityMatrix[26];
rot2Mat26[[1,1]]=1/Sqrt[2];
rot2Mat26[[18,18]]=1/Sqrt[2];
invRot2Mat26=Inverse[rot2Mat26];
f4OrthonormalBase=Table[Simplify[rot2Mat26.f4OrthonormalBase[[i]].invRot2Mat26],{i,52}];
```

### B.2 Basis for $\mathfrak{f}_4$

The basis for $\mathfrak{f}_4$ we obtained is

$$
\begin{aligned}
T_1 &= -\frac{1}{2}M_{1\,19} + \frac{1}{2}M_{2\,10} + \frac{1}{2}M_{3\,11} + \frac{1}{2}M_{4\,12} + \frac{1}{2}M_{5\,13} + \frac{1}{2}M_{6\,14} + \frac{1}{2}M_{7\,15} + \frac{1}{2}M_{8\,16} + \frac{1}{2}M_{9\,17} + \frac{\sqrt{3}}{2}M_{18\,19},\\
T_2 &= -\frac{1}{2}M_{1\,20} + \frac{1}{2}M_{2\,11} - \frac{1}{2}M_{3\,10} - \frac{1}{2}M_{4\,14} - \frac{1}{2}M_{5\,17} + \frac{1}{2}M_{6\,12} - \frac{1}{2}M_{7\,16} + \frac{1}{2}M_{8\,15} + \frac{1}{2}M_{9\,13} + \frac{\sqrt{3}}{2}M_{18\,20},\\
T_3 &= -\frac{1}{2}M_{1\,21} + \frac{1}{2}M_{2\,12} + \frac{1}{2}M_{3\,14} - \frac{1}{2}M_{4\,10} - \frac{1}{2}M_{5\,15} - \frac{1}{2}M_{6\,11} + \frac{1}{2}M_{7\,13} - \frac{1}{2}M_{8\,17} + \frac{1}{2}M_{9\,16} + \frac{\sqrt{3}}{2}M_{18\,21},\\
T_4 &= -\frac{1}{2}M_{1\,22} + \frac{1}{2}M_{2\,13} + \frac{1}{2}M_{3\,17} + \frac{1}{2}M_{4\,15} - \frac{1}{2}M_{5\,10} - \frac{1}{2}M_{6\,16} - \frac{1}{2}M_{7\,12} + \frac{1}{2}M_{8\,14} - \frac{1}{2}M_{9\,11} + \frac{\sqrt{3}}{2}M_{18\,22},\\
T_5 &= -\frac{1}{2}M_{1\,23} + \frac{1}{2}M_{2\,14} - \frac{1}{2}M_{3\,12} + \frac{1}{2}M_{4\,11} + \frac{1}{2}M_{5\,16} - \frac{1}{2}M_{6\,10} - \frac{1}{2}M_{7\,17} - \frac{1}{2}M_{8\,13} + \frac{1}{2}M_{9\,15} + \frac{\sqrt{3}}{2}M_{18\,23},\\
T_6 &= -\frac{1}{2}M_{1\,24} + \frac{1}{2}M_{2\,15} + \frac{1}{2}M_{3\,16} - \frac{1}{2}M_{4\,13} + \frac{1}{2}M_{5\,12} + \frac{1}{2}M_{6\,17} - \frac{1}{2}M_{7\,10} - \frac{1}{2}M_{8\,11} - \frac{1}{2}M_{9\,14} + \frac{\sqrt{3}}{2}M_{18\,24},\\
T_7 &= -\frac{1}{2}M_{1\,25} + \frac{1}{2}M_{2\,16} - \frac{1}{2}M_{3\,15} + \frac{1}{2}M_{4\,17} - \frac{1}{2}M_{5\,14} + \frac{1}{2}M_{6\,13} + \frac{1}{2}M_{7\,11} - \frac{1}{2}M_{8\,10} - \frac{1}{2}M_{9\,12} + \frac{\sqrt{3}}{2}M_{18\,25},\\
T_8 &= -\frac{1}{2}M_{1\,26} + \frac{1}{2}M_{2\,17} - \frac{1}{2}M_{3\,13} - \frac{1}{2}M_{4\,16} + \frac{1}{2}M_{5\,11} - \frac{1}{2}M_{6\,15} + \frac{1}{2}M_{7\,14} + \frac{1}{2}M_{8\,12} - \frac{1}{2}M_{9\,10} + \frac{\sqrt{3}}{2}M_{18\,26},\\
T_9 &= M_{8\,9} + \frac{1}{2}M_{10\,12} + \frac{1}{2}M_{11\,14} - \frac{1}{2}M_{13\,15} + \frac{1}{2}M_{16\,17} + \frac{1}{2}M_{19\,21} + \frac{1}{2}M_{20\,23} - \frac{1}{2}M_{22\,24} + \frac{1}{2}M_{25\,26},\\
T_{10} &= M_{7\,9} + \frac{1}{2}M_{10\,14} - \frac{1}{2}M_{11\,12} + \frac{1}{2}M_{13\,16} + \frac{1}{2}M_{15\,17} + \frac{1}{2}M_{19\,23} - \frac{1}{2}M_{20\,21} + \frac{1}{2}M_{22\,25} + \frac{1}{2}M_{24\,26},\\
T_{11} &= M_{7\,8} + \frac{1}{2}M_{10\,11} - \frac{1}{2}M_{12\,14} - \frac{1}{2}M_{13\,17} + \frac{1}{2}M_{15\,16} + \frac{1}{2}M_{19\,20} - \frac{1}{2}M_{21\,23} - \frac{1}{2}M_{22\,26} + \frac{1}{2}M_{24\,25},\\
T_{12} &= M_{6\,9} - \frac{1}{2}M_{10\,15} - \frac{1}{2}M_{11\,16} + \frac{1}{2}M_{12\,13} + \frac{1}{2}M_{14\,17} - \frac{1}{2}M_{19\,24} - \frac{1}{2}M_{20\,25} + \frac{1}{2}M_{21\,22} + \frac{1}{2}M_{23\,26},\\
T_{13} &= M_{6\,8} + \frac{1}{2}M_{10\,13} + \frac{1}{2}M_{11\,17} + \frac{1}{2}M_{12\,15} + \frac{1}{2}M_{14\,16} + \frac{1}{2}M_{19\,22} + \frac{1}{2}M_{20\,26} + \frac{1}{2}M_{21\,24} + \frac{1}{2}M_{23\,25},\\
T_{14} &= M_{6\,7} + \frac{1}{2}M_{10\,17} - \frac{1}{2}M_{11\,13} - \frac{1}{2}M_{12\,16} + \frac{1}{2}M_{14\,15} + \frac{1}{2}M_{19\,26} - \frac{1}{2}M_{20\,22} - \frac{1}{2}M_{21\,25} + \frac{1}{2}M_{23\,24},\\
T_{15} &= M_{5\,9} + \frac{1}{2}M_{10\,11} - \frac{1}{2}M_{12\,14} + \frac{1}{2}M_{13\,17} - \frac{1}{2}M_{15\,16} + \frac{1}{2}M_{19\,20} - \frac{1}{2}M_{21\,23} + \frac{1}{2}M_{22\,26} - \frac{1}{2}M_{24\,25},\\
T_{16} &= M_{5\,8} - \frac{1}{2}M_{10\,14} + \frac{1}{2}M_{11\,12} + \frac{1}{2}M_{13\,16} + \frac{1}{2}M_{15\,17} - \frac{1}{2}M_{19\,23} + \frac{1}{2}M_{20\,21} + \frac{1}{2}M_{22\,25} + \frac{1}{2}M_{24\,26},\\
T_{17} &= M_{5\,7} + \frac{1}{2}M_{10\,12} + \frac{1}{2}M_{11\,14} + \frac{1}{2}M_{13\,15} - \frac{1}{2}M_{16\,17} + \frac{1}{2}M_{19\,21} + \frac{1}{2}M_{20\,23} + \frac{1}{2}M_{22\,24} - \frac{1}{2}M_{25\,26},\\
T_{18} &= M_{5\,6} + \frac{1}{2}M_{10\,16} - \frac{1}{2}M_{11\,15} + \frac{1}{2}M_{12\,17} + \frac{1}{2}M_{13\,14} + \frac{1}{2}M_{19\,25} - \frac{1}{2}M_{20\,24} + \frac{1}{2}M_{21\,26} + \frac{1}{2}M_{22\,23},\\
T_{19} &= M_{4\,9} - \frac{1}{2}M_{10\,16} + \frac{1}{2}M_{11\,15} + \frac{1}{2}M_{12\,17} + \frac{1}{2}M_{13\,14} - \frac{1}{2}M_{19\,25} + \frac{1}{2}M_{20\,24} + \frac{1}{2}M_{21\,26} + \frac{1}{2}M_{22\,23},\\
T_{20} &= M_{4\,8} + \frac{1}{2}M_{10\,17} - \frac{1}{2}M_{11\,13} + \frac{1}{2}M_{12\,16} - \frac{1}{2}M_{14\,15} + \frac{1}{2}M_{19\,26} - \frac{1}{2}M_{20\,22} + \frac{1}{2}M_{21\,25} - \frac{1}{2}M_{23\,24},\\
T_{21} &= M_{4\,7} - \frac{1}{2}M_{10\,13} - \frac{1}{2}M_{11\,17} + \frac{1}{2}M_{12\,15} + \frac{1}{2}M_{14\,16} - \frac{1}{2}M_{19\,22} - \frac{1}{2}M_{20\,26} + \frac{1}{2}M_{21\,24} + \frac{1}{2}M_{23\,25},\\
T_{22} &= M_{4\,6} + \frac{1}{2}M_{10\,11} + \frac{1}{2}M_{12\,14} - \frac{1}{2}M_{13\,17} - \frac{1}{2}M_{15\,16} + \frac{1}{2}M_{19\,20} + \frac{1}{2}M_{21\,23} - \frac{1}{2}M_{22\,26} - \frac{1}{2}M_{24\,25},\\
T_{23} &= M_{4\,5} + \frac{1}{2}M_{10\,15} + \frac{1}{2}M_{11\,16} + \frac{1}{2}M_{12\,13} + \frac{1}{2}M_{14\,17} + \frac{1}{2}M_{19\,24} + \frac{1}{2}M_{20\,25} + \frac{1}{2}M_{21\,22} + \frac{1}{2}M_{23\,26},\\
T_{24} &= M_{3\,9} - \frac{1}{2}M_{10\,13} + \frac{1}{2}M_{11\,17} - \frac{1}{2}M_{12\,15} + \frac{1}{2}M_{14\,16} - \frac{1}{2}M_{19\,22} + \frac{1}{2}M_{20\,26} - \frac{1}{2}M_{21\,24} + \frac{1}{2}M_{23\,25},\\
T_{25} &= M_{3\,8} - \frac{1}{2}M_{10\,15} + \frac{1}{2}M_{11\,16} + \frac{1}{2}M_{12\,13} - \frac{1}{2}M_{14\,17} - \frac{1}{2}M_{19\,24} + \frac{1}{2}M_{20\,25} + \frac{1}{2}M_{21\,22} - \frac{1}{2}M_{23\,26},
\end{aligned}
$$

$$T_{26} = M_{3\,7} + \frac{1}{2}M_{10\,16} + \frac{1}{2}M_{11\,15} + \frac{1}{2}M_{12\,17} - \frac{1}{2}M_{13\,14} + \frac{1}{2}M_{19\,25} + \frac{1}{2}M_{20\,24} + \frac{1}{2}M_{21\,26} - \frac{1}{2}M_{22\,23},$$
$$T_{27} = M_{3\,6} - \frac{1}{2}M_{10\,12} + \frac{1}{2}M_{11\,14} + \frac{1}{2}M_{13\,15} + \frac{1}{2}M_{16\,17} - \frac{1}{2}M_{19\,21} + \frac{1}{2}M_{20\,23} + \frac{1}{2}M_{22\,24} + \frac{1}{2}M_{25\,26},$$
$$T_{28} = M_{3\,5} + \frac{1}{2}M_{10\,17} + \frac{1}{2}M_{11\,13} - \frac{1}{2}M_{12\,16} - \frac{1}{2}M_{14\,15} + \frac{1}{2}M_{19\,26} + \frac{1}{2}M_{20\,22} - \frac{1}{2}M_{21\,25} - \frac{1}{2}M_{23\,24},$$
$$T_{29} = M_{3\,4} + \frac{1}{2}M_{10\,14} + \frac{1}{2}M_{11\,12} + \frac{1}{2}M_{13\,16} - \frac{1}{2}M_{15\,17} + \frac{1}{2}M_{19\,23} + \frac{1}{2}M_{20\,21} + \frac{1}{2}M_{22\,25} - \frac{1}{2}M_{24\,26},$$
$$T_{30} = M_{2\,9} + \frac{1}{2}M_{10\,17} + \frac{1}{2}M_{11\,13} + \frac{1}{2}M_{12\,16} + \frac{1}{2}M_{14\,15} - \frac{1}{2}M_{19\,26} - \frac{1}{2}M_{20\,22} - \frac{1}{2}M_{21\,25} - \frac{1}{2}M_{23\,24},$$
$$T_{31} = M_{2\,8} + \frac{1}{2}M_{10\,16} + \frac{1}{2}M_{11\,15} - \frac{1}{2}M_{12\,17} + \frac{1}{2}M_{13\,14} - \frac{1}{2}M_{19\,25} - \frac{1}{2}M_{20\,24} + \frac{1}{2}M_{21\,26} - \frac{1}{2}M_{22\,23},$$
$$T_{32} = M_{2\,7} + \frac{1}{2}M_{10\,15} - \frac{1}{2}M_{11\,16} + \frac{1}{2}M_{12\,13} - \frac{1}{2}M_{14\,17} - \frac{1}{2}M_{19\,24} + \frac{1}{2}M_{20\,25} - \frac{1}{2}M_{21\,22} + \frac{1}{2}M_{23\,26},$$
$$T_{33} = M_{2\,6} + \frac{1}{2}M_{10\,14} + \frac{1}{2}M_{11\,12} - \frac{1}{2}M_{13\,16} + \frac{1}{2}M_{15\,17} - \frac{1}{2}M_{19\,23} - \frac{1}{2}M_{20\,21} + \frac{1}{2}M_{22\,25} - \frac{1}{2}M_{24\,26},$$
$$T_{34} = M_{2\,5} + \frac{1}{2}M_{10\,13} - \frac{1}{2}M_{11\,17} - \frac{1}{2}M_{12\,15} + \frac{1}{2}M_{14\,16} - \frac{1}{2}M_{19\,22} + \frac{1}{2}M_{20\,26} + \frac{1}{2}M_{21\,24} - \frac{1}{2}M_{23\,25},$$
$$T_{35} = M_{2\,4} + \frac{1}{2}M_{10\,12} - \frac{1}{2}M_{11\,14} + \frac{1}{2}M_{13\,15} + \frac{1}{2}M_{16\,17} - \frac{1}{2}M_{19\,21} + \frac{1}{2}M_{20\,23} - \frac{1}{2}M_{22\,24} - \frac{1}{2}M_{25\,26},$$
$$T_{36} = M_{2\,3} + \frac{1}{2}M_{10\,11} + \frac{1}{2}M_{12\,14} + \frac{1}{2}M_{13\,17} + \frac{1}{2}M_{15\,16} - \frac{1}{2}M_{19\,20} - \frac{1}{2}M_{21\,23} - \frac{1}{2}M_{22\,26} - \frac{1}{2}M_{24\,25},$$
$$T_{37} = \frac{1}{2}M_{1\,17} + \frac{1}{2}M_{2\,26} + \frac{1}{2}M_{3\,22} + \frac{1}{2}M_{4\,25} - \frac{1}{2}M_{5\,20} + \frac{1}{2}M_{6\,24} - \frac{1}{2}M_{7\,23} - \frac{1}{2}M_{8\,21} + \frac{1}{2}M_{9\,19} - \frac{\sqrt{3}}{2}M_{17\,18},$$
$$T_{38} = \frac{1}{2}M_{1\,16} + \frac{1}{2}M_{2\,25} + \frac{1}{2}M_{3\,24} - \frac{1}{2}M_{4\,26} + \frac{1}{2}M_{5\,23} - \frac{1}{2}M_{6\,22} - \frac{1}{2}M_{7\,20} + \frac{1}{2}M_{8\,19} + \frac{1}{2}M_{9\,21} - \frac{\sqrt{3}}{2}M_{16\,18},$$
$$T_{39} = \frac{1}{2}M_{1\,15} + \frac{1}{2}M_{2\,24} - \frac{1}{2}M_{3\,25} + \frac{1}{2}M_{4\,22} - \frac{1}{2}M_{5\,21} - \frac{1}{2}M_{6\,26} + \frac{1}{2}M_{7\,19} + \frac{1}{2}M_{8\,20} + \frac{1}{2}M_{9\,23} - \frac{\sqrt{3}}{2}M_{15\,18},$$
$$T_{40} = \frac{1}{2}M_{1\,14} + \frac{1}{2}M_{2\,23} + \frac{1}{2}M_{3\,21} - \frac{1}{2}M_{4\,20} - \frac{1}{2}M_{5\,25} + \frac{1}{2}M_{6\,19} + \frac{1}{2}M_{7\,26} + \frac{1}{2}M_{8\,22} - \frac{1}{2}M_{9\,24} - \frac{\sqrt{3}}{2}M_{14\,18},$$
$$T_{41} = \frac{1}{2}M_{1\,13} + \frac{1}{2}M_{2\,22} - \frac{1}{2}M_{3\,26} - \frac{1}{2}M_{4\,24} + \frac{1}{2}M_{5\,19} + \frac{1}{2}M_{6\,25} + \frac{1}{2}M_{7\,21} - \frac{1}{2}M_{8\,23} + \frac{1}{2}M_{9\,20} - \frac{\sqrt{3}}{2}M_{13\,18},$$
$$T_{42} = \frac{1}{2}M_{1\,12} + \frac{1}{2}M_{2\,21} - \frac{1}{2}M_{3\,23} + \frac{1}{2}M_{4\,19} + \frac{1}{2}M_{5\,24} + \frac{1}{2}M_{6\,20} - \frac{1}{2}M_{7\,22} + \frac{1}{2}M_{8\,26} - \frac{1}{2}M_{9\,25} - \frac{\sqrt{3}}{2}M_{12\,18},$$
$$T_{43} = \frac{1}{2}M_{1\,11} + \frac{1}{2}M_{2\,20} + \frac{1}{2}M_{3\,19} + \frac{1}{2}M_{4\,23} + \frac{1}{2}M_{5\,26} - \frac{1}{2}M_{6\,21} + \frac{1}{2}M_{7\,25} - \frac{1}{2}M_{8\,24} - \frac{1}{2}M_{9\,22} - \frac{\sqrt{3}}{2}M_{11\,18},$$
$$T_{44} = \frac{1}{2}M_{1\,10} + \frac{1}{2}M_{2\,19} - \frac{1}{2}M_{3\,20} - \frac{1}{2}M_{4\,21} - \frac{1}{2}M_{5\,22} - \frac{1}{2}M_{6\,23} - \frac{1}{2}M_{7\,24} - \frac{1}{2}M_{8\,25} - \frac{1}{2}M_{9\,26} - \frac{\sqrt{3}}{2}M_{10\,18},$$
$$T_{45} = M_{1\,9} - \frac{1}{2}M_{10\,26} - \frac{1}{2}M_{11\,22} - \frac{1}{2}M_{12\,25} + \frac{1}{2}M_{13\,20} - \frac{1}{2}M_{14\,24} + \frac{1}{2}M_{15\,23} + \frac{1}{2}M_{16\,21} + \frac{1}{2}M_{17\,19},$$
$$T_{46} = M_{1\,8} - \frac{1}{2}M_{10\,25} - \frac{1}{2}M_{11\,24} + \frac{1}{2}M_{12\,26} - \frac{1}{2}M_{13\,23} + \frac{1}{2}M_{14\,22} + \frac{1}{2}M_{15\,20} + \frac{1}{2}M_{16\,19} - \frac{1}{2}M_{17\,21},$$
$$T_{47} = M_{1\,7} - \frac{1}{2}M_{10\,24} + \frac{1}{2}M_{11\,25} - \frac{1}{2}M_{12\,22} + \frac{1}{2}M_{13\,21} + \frac{1}{2}M_{14\,26} + \frac{1}{2}M_{15\,19} - \frac{1}{2}M_{16\,20} - \frac{1}{2}M_{17\,23},$$
$$T_{48} = M_{1\,6} - \frac{1}{2}M_{10\,23} - \frac{1}{2}M_{11\,21} + \frac{1}{2}M_{12\,20} + \frac{1}{2}M_{13\,25} + \frac{1}{2}M_{14\,19} - \frac{1}{2}M_{15\,26} - \frac{1}{2}M_{16\,22} + \frac{1}{2}M_{17\,24},$$
$$T_{49} = M_{1\,5} - \frac{1}{2}M_{10\,22} + \frac{1}{2}M_{11\,26} + \frac{1}{2}M_{12\,24} + \frac{1}{2}M_{13\,19} - \frac{1}{2}M_{14\,25} - \frac{1}{2}M_{15\,21} + \frac{1}{2}M_{16\,23} - \frac{1}{2}M_{17\,20},$$
$$T_{50} = M_{1\,4} - \frac{1}{2}M_{10\,21} + \frac{1}{2}M_{11\,23} + \frac{1}{2}M_{12\,19} - \frac{1}{2}M_{13\,24} - \frac{1}{2}M_{14\,20} + \frac{1}{2}M_{15\,22} - \frac{1}{2}M_{16\,26} + \frac{1}{2}M_{17\,25},$$
$$T_{51} = M_{1\,3} - \frac{1}{2}M_{10\,20} + \frac{1}{2}M_{11\,19} - \frac{1}{2}M_{12\,23} - \frac{1}{2}M_{13\,26} + \frac{1}{2}M_{14\,21} - \frac{1}{2}M_{15\,25} + \frac{1}{2}M_{16\,24} + \frac{1}{2}M_{17\,22},$$
$$T_{52} = M_{1\,2} + \frac{1}{2}M_{10\,19} + \frac{1}{2}M_{11\,20} + \frac{1}{2}M_{12\,21} + \frac{1}{2}M_{13\,22} + \frac{1}{2}M_{14\,23} + \frac{1}{2}M_{15\,24} + \frac{1}{2}M_{16\,25} + \frac{1}{2}M_{17\,26},$$

where $M_{ij} = E_{ij} - E_{ji}$ are generators of $\mathfrak{so}\,(26)$.

### B.3 Solution of the Yang-Baxter equation

```
tensI=Table[KroneckerDelta[k1,i1]*KroneckerDelta[k2,i2],{k1,26},{k2,26},{i1,26},{i2,26}];
tensP=Table[KroneckerDelta[k1,i2]*KroneckerDelta[k2,i1],{k1,26},{k2,26},{i1,26},{i2,26}];
tensK=Table[KroneckerDelta[k1,k2]*KroneckerDelta[i1,i2],{k1,26},{k2,26},{i1,26},{i2,26}];
tensC=Table[Sum[f4OrthonormalBase[[a,k1,i1]]*f4OrthonormalBase[[a,k2,i2]],{a,52}],
        {k1,26},{k2,26},{i1,26},{i2,26}];
tensS=Table[(tensC[[k1,k2,i1,i2]]+tensC[[k2,k1,i1,i2]])/2,{k1,26},{k2,26},{i1,26},{i2,26}];
tensAs=Table[(tensC[[k1,k2,i1,i2]]-tensC[[k2,k1,i1,i2]])/2,{k1,26},{k2,26},{i1,26},{i2,26}];

params={a1,a2,c1,c2,c3,d1,d2,p1,p2,p3};
tensR[x_][k1_,k2_,i1_,i2_]:=tensR[x][k1,k2,i1,i2]=
        tensI[[k1,k2,i1,i2]]*(-x+a1*x/(x-p1))+
        tensP[[k1,k2,i1,i2]]*(1+a2*x/(x-p1))+
        tensK[[k1,k2,i1,i2]]*(c1*x/(x-p1)+c2*x/(x-p2)+c3*x/(x-p3))+
        tensS[[k1,k2,i1,i2]]*(d1*x/(x-p1))+
        tensAs[[k1,k2,i1,i2]]*(d2*x/(x-p2));

hasNoTensorsQ[s_]:=MemberQ[s,d1->0]||MemberQ[s,d2->0]||MemberQ[s,c3->0];
hasZeroPolesQ[s_]:=MemberQ[s,p1->0]||MemberQ[s,p2->0]||MemberQ[s,p3->0];
hasSamePolesQ[s_]:=MemberQ[s,p1->p2]||MemberQ[s,p1->p3]||
        MemberQ[s,p2->p1]||MemberQ[s,p2->p3]||MemberQ[s,p3->p1]||MemberQ[s,p3->p2];
goodSolnQ[s_]:=!(hasNoTensorsQ[s]||hasZeroPolesQ[s]||hasSamePolesQ[s]);

eqnYB[k1_,k2_,k3_,i1_,i2_,i3_]:=
        Sum[tensR[u][k1,k2,j1,j2]*tensR[u+v][j1,k3,i1,j3]*tensR[v][j2,j3,i2,i3]-
        tensR[v][k2,k3,j2,j3]*tensR[u+v][k1,j3,j1,i3]*tensR[u][j1,j2,i1,i2],{j1,26},{j2,26},{j3,26}];

coeffs1={};
Table[coeffs1=Union[coeffs1,
        Replace[CoefficientRules[Numerator[Together[eqnYB[k1,k2,k3,i1,i2,i3]]],{u,v}],
        {Rule[List[u_,v_],c_]->c},2]];,
        {k1,26},{k2,26},{k3,3},{i1,1},{i2,1},{i3,1}];
coeffs1=Union[Table[Expand[coeffs1[[i]]/coeffs1[[i,1,1]]],{i,Length[coeffs1]}]];
coeffs1=SortBy[coeffs1,Length[#]&];
solns1=Select[Simplify[Solve[coeffs1[[;;9]]==0,params]],goodSolnQ[#]&];
solnPart[1]={a1->-5*d1/2-p1,a2->-1,d2->d1};

coeffs2=coeffs1/.solnPart[1];
solnPart[2]=Solve[coeffs2[[1]]==0,{c3}][[1]];

coeffs3=Sort[coeffs2/.solnPart[2]];
solnPart[3]=Select[Solve[coeffs3[[;;35]]==0,{c1,c2,p1,p2,p3,d1}],goodSolnQ[#]&][[1]];

soln=solnPart[1];
For[i=2,i<=3,i++,
        soln=Join[soln/.solnPart[i],solnPart[i]];];

reqUnitary[mu_,k1_,k2_,i1_,i2_]:=
        Sum[tensR[u][k1,k2,j1,j2]*tensR[-u][j1,j2,i1,i2],{j1,26},{j2,26}]-
        mu[u]*KroneckerDelta[k1,i1]*KroneckerDelta[k2,i2];
eqnUnitary=Union[Flatten[Table[(reqUnitary[mu,i1,i2,k1,k2]/.soln)==0,
        {i1,3},{i2,3},{k1,3},{k2,3}]]];
solnUnitary=Solve[eqnUnitary,mu[u]];
```

## C Solution with $E_6$ symmetry

### C.1 Construction of a basis for $\mathfrak{e}_6$

```
octBaseNames=Array[e,8];
octBase=IdentityMatrix[8];
```

```
octListForm[x_]:=Coefficient[x,octBaseNames];
octMultiplicationTable={
        {e[1],e[2],e[3],e[4],e[5],e[6],e[7],e[8]},
        {e[2],-e[1],e[5],e[8],-e[3],e[7],-e[6],-e[4]},
        {e[3],-e[5],-e[1],e[6],e[2],-e[4],e[8],-e[7]},
        {e[4],-e[8],-e[6],-e[1],e[7],e[3],-e[5],e[2]},
        {e[5],e[3],-e[2],-e[7],-e[1],e[8],e[4],-e[6]},
        {e[6],-e[7],e[4],-e[3],-e[8],-e[1],e[2],e[5]},
        {e[7],e[6],-e[8],e[5],-e[4],-e[2],-e[1],e[3]},
        {e[8],e[4],e[7],-e[2],e[6],-e[5],-e[3],-e[1]}};
octProduct[x_,y_]:=octListForm[Sum[x[[i]]*y[[j]]*octMultiplicationTable[[i,j]],{i,8},{j,8}]];
octConjugate[x_]:=Join[{x[[1]]},-x[[2;;8]]];
octNormSquare[x_]:=Sum[x[[i]]^2,{i,8}];

jordanToR27[AA_]:=Join[
        {AA[[1,1,1]]},AA[[1,2,1;;8]],AA[[1,3,1;;8]],{AA[[2,2,1]]},AA[[2,3,1;;8]],{AA[[3,3,1]]}];
r27ToJordan[vv_]:=Module[{a1,a2,a3,o1,o2,o3,o1c,o2c,o3c},
        a1=a2=a3=ConstantArray[0,8];
        a1[[1]]=vv[[1]];a2[[1]]=vv[[18]];a3[[1]]=vv[[27]];
        o1=vv[[2;;9]];o2=vv[[10;;17]];o3=vv[[19;;26]];
        o1c=octConjugate[o1];o2c=octConjugate[o2];o3c=octConjugate[o3];
        {{a1,o1,o2},{o1c,a2,o3},{o2c,o3c,a3}}];
octMatrProduct[a_,b_]:=Table[Sum[octProduct[a[[i,k]],b[[k,j]]],{k,3}],{i,3},{j,3}];
jordanProduct[a_,b_]:=1/2(octMatrProduct[a,b]+octMatrProduct[b,a]);
r27Product[x_,y_]:=jordanToR27[jordanProduct[r27ToJordan[x],r27ToJordan[y]]];

X=Array[x,27];
Y=Join[Array[y,26],{-y[1]-y[18]}];
XY=r27Product[X,Y];

RYGeneralForm=Table[Coefficient[XY[[i]],x[j]],{i,27},{j,27}];

ruleRenameParams=Table[y[j]->KroneckerDelta[i,j],{i,26},{j,26}];
RYStandardBase=RYGeneralForm/.ruleRenameParams;

killingForm[a_,b_]:=Sum[a[[i,j]]*b[[j,i]],{i,27},{j,27}];
killingNormSquare[a_]:=killingForm[a,a];
killingProject[a_,b_]:=killingForm[a,b]/killingForm[b,b]*b;
RYOrthogonalBase={};
Table[AppendTo[RYOrthogonalBase,RYStandardBase[[i]]-
        Sum[killingProject[RYStandardBase[[i]],RYOrthogonalBase[[j]]],{j,i-1}]];,
        {i,Length[RYStandardBase]}];

RYOrthonormalBase=Table[
        Sqrt[6/killingNormSquare[RYOrthogonalBase[[i]]]]*RYOrthogonalBase[[i]],{i,26}];
rot1=EE=IdentityMatrix[27];
rot1[[1]]=(EE[[1]]-EE[[18]])/Sqrt[2];
rot1[[18]]=(EE[[1]]+EE[[18]]-2*EE[[27]])/Sqrt[6];
rot1[[27]]=(EE[[1]]+EE[[18]]+EE[[27]])/Sqrt[3];
invRot1=Transpose[rot1];
rot2=IdentityMatrix[27];
rot2[[1,1]]=rot2[[18,18]]=1/Sqrt[2];
invRot2=Inverse[rot2];
RYOrthonormalBase=Table[Simplify[rot2.rot1.RYOrthonormalBase[[i]].invRot1.invRot2*I],{i,26}];
```