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Bogoliubov Laboratory for Theoretical Physics

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A coupled fermion-kink system in the ϕ^4-model

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1 Introduction

Topological solitons has gained an increasing interest for the last decades. Many models which support classical soliton solutions, have been intensively studied in a wide variety of physical contexts. The ϕ^4-model arises in many different physical situations, it is a prototype for many non-linear systems. Indeed, this model is known in the cosmological context, condensed matter physics. In particular, it was applied to describe solitary waves in shape-memory alloys, it also can be used as a phenomenological theory of the non-perturbative transitions in a polyacetylene chain. Furthermore, the ϕ^4-model has been applied in biophysics to describe soliton excitations in DNA double helices.

As a result of the interaction of fermions with an inhomogeneous background bosonic field of a soliton, fermions can be localized on it, thereby forming a spectrum of localized bound states. This fact has been known since the 70s of the 20th century by the example of topological solitons such as kinks and domain walls [1]. In more detail, the bound states of fermions on kinks and domain walls were discussed in [2]-[6].

At the moment there is no exact analytical solution to the problem of the interaction of the kink with a fermion. The reason for this is the self-consistence of the problem: a fermion localizes on the kink, but distorts the profile of the latter. Note that this distortion depends on the coupling constant, as a result of which the distortion, generally speaking, is not small. This suggests that the system is self-consistent and the perturbation theory is not applicable to it. An attempt to take into account the backreaction of a fermion to a kink was made in [2], but some of the assumptions used in this paper are doubtful.In [3]-[6] the backreaction is neglected.

The main purpose of this work is to investigate the effects of backreaction of the fermionic modes coupled with the kink of the ϕ^4-model. In other words, the aims are to find fermionic modes localized on the kink, the kink profile distortion and to examine how solutions change with changing system parameters: a coupling constant and a fermionic mass.

2 The Model

The Lagrangian density of the coupled fermion-kink system in 1+1 dimensions can be written as

$$\mathcal{L} = \mathcal{L}_\phi + \mathcal{L}_\Psi, \tag{1}$$

where $\mathcal{L}_\phi$ is a Lagrangian of the real scalar field ϕ:

$$\mathcal{L}_\phi = \frac{1}{2}(\partial_\mu \phi)^2 - V(\phi). \tag{2}$$

Here we are using the flat metric $g^{\mu\nu} = \mathrm{diag}(1, -1)$, $\mu = \{0, 1\}$. The scalar field $\phi(x, t)$ is dimensionless: $[\phi] = 1$. This Lagrangian contains a potential term, which takes the form

$$V(\phi) = \frac{\mu^2}{2}(\phi^2 - 1)^2 \tag{3}$$

in the ϕ^4 model, μ is a mass parameter.

The Lagrangian for fermions coupled to the scalar field is given by

$$\mathcal{L}_\Psi = \overline{\Psi} i \widehat{\partial} \Psi - m \overline{\Psi} \Psi - g \overline{\Psi} \Psi \phi, \tag{4}$$

where g is a coupling constant, $[g] = 1$, m is a fermion mass parameter, $[\mu] = [m]$. Note that the fermion field Ψ is a two-component spinor: $\Psi \equiv \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}$, $[\Psi] = [m]^{1/2}$. Here we use the following notations:

$$\widehat{\partial}\Psi \equiv \gamma^\mu \partial_\mu \Psi, \qquad\qquad \overline{\Psi} \equiv \Psi \gamma^0,$$

where γ_μ are the Dirac matrices. We use the following representation of Dirac matrices:

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad\qquad \gamma^1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}. \tag{5}$$

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For further convenience, we use dimensionless coordinates $\tilde{x} = \mu x$, $\tilde{t} = \mu t$, fermionic field $\tilde{\Psi} = \mu^{-1/2}\Psi$, coupling constant $\tilde{g} = \mu^{-1} g$ and fermion mass $\tilde{m} = \mu^{-1} m$. Therefore, the Lagrangian can be written as $\mathcal{L}(x, t) = \mu^2 \tilde{\mathcal{L}}(\tilde{x}, \tilde{t})$, where

$$\tilde{\mathcal{L}}(\tilde{x}, \tilde{t}) = \frac{1}{2}(\tilde{\partial}_\mu \phi)^2 - \frac{1}{2}(\phi^2 - 1)^2 + \tilde{\overline{\Psi}} i \tilde{\widehat{\partial}} \tilde{\Psi} - \tilde{m} \tilde{\overline{\Psi}} \tilde{\Psi} - \tilde{g} \tilde{\overline{\Psi}} \tilde{\Psi} \phi.$$

Hereinafter we omit everywhere the tilde and work in terms of dimensionless fields, coupling constant, space-time variables and a fermionic mass. So, due to the Lagrangian equivalence property the action of the system is given by

$$S = \int d^2x\, \mathcal{L} = \int d^2x \left[\frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}(\phi^2 - 1)^2 + \overline{\Psi} i \widehat{\partial} \Psi - m \overline{\Psi}\Psi - g \overline{\Psi}\Psi\phi \right]. \quad (6)$$

The principle of least action gives us equations of motion:

$$\begin{cases} \partial_\mu \partial_\mu \phi - 2\phi + 2\phi^3 + g\overline{\Psi}\Psi = 0, \\ i\gamma^\mu \partial_\mu \Psi - m\Psi - g\Psi\phi = 0. \end{cases} \quad (7)$$

In the case of non-interacting fields, i.e. $g = 0$, the equation of motion for the scalar field takes the form

$$\partial_\mu \partial_\mu \phi - 2\phi + 2\phi^3 = 0. \quad (8)$$

It can be easily shown that this scalar field model has two vacuum states $\phi = \pm 1$ corresponding to the degenerate absolute minimum of the energy of the system. The solution $\phi = 0$ corresponds to the unstable vacuum without spontaneous symmetry breaking. Another static solution of (8) with finite energy is a topological soliton called "kink":

$$\phi_K(x, t) = \tanh{(x - x_0)}. \quad (9)$$

The properties and characteristics of this solution are widely known, see [7]-[8].

Let us discuss the fermionic sector of the theory. In the case of stationary states

$$\Psi(x, t) = e^{-i\varepsilon t} \psi_\varepsilon(x) = e^{-i\varepsilon t} \begin{pmatrix} u_\varepsilon(x) \\ v_\varepsilon(x) \end{pmatrix}, \quad (10)$$

and fermionic equation of motion takes the form

$$\begin{aligned} \frac{du_\varepsilon}{dx} + m u_\varepsilon + g\phi(x) u_\varepsilon &= \varepsilon v_\varepsilon, \\ -\frac{dv_\varepsilon}{dx} + m v_\varepsilon + g\phi(x) v_\varepsilon &= \varepsilon u_\varepsilon. \end{aligned} \quad (11)$$

Note that we should take into account the normalization condition

$$\int\limits_{-\infty}^{\infty} \left[u_\varepsilon(x)^2 + v_\varepsilon(x)^2 \right] dx = 1. \quad (12)$$

While absence of interaction between the scalar and fermion field ($g = 0$) the equations of the system (7) become independent: the presence of fermions does not distort the profile of the kink. But if $g \neq 0$ the kink distortion becomes stronger with increasing g. Thus, the problem should be solved in self-consistent case. An exception is the so called zero mode – the lowest energy state of a fermion. It can be easily shown that $\varepsilon = 0$ is eigenvalue of (7) with eigenvector (we consider $x_0 = 0$)

$$\Psi^g_{\varepsilon=0}(x,t) = \mathcal{N}(g,m) \begin{pmatrix} \dfrac{e^{-mx}}{\cosh^g x} \\ \\ 0 \end{pmatrix}, \tag{13}$$

where $\mathcal{N}(g,m)$ can be found from the normalization condition (12).

3 Numerical results

3.1 A system of equations

As far as we know, at present there is no exact analytical solution for nonzero modes. However, this fact does not forbid us to solve the system numerically. The whole system of equations and conditions takes the form

$$\begin{cases} \dfrac{d^2\phi}{dx^2} + 2\phi - 2\phi^3 - 2g u_\varepsilon v_\varepsilon = 0 \\ \\ \dfrac{du_\varepsilon}{dx} + m u_\varepsilon + g\phi(x) u_\varepsilon = \varepsilon v_\varepsilon \\ \\ -\dfrac{dv_\varepsilon}{dx} + m v_\varepsilon + g\phi(x) v_\varepsilon = \varepsilon u_\varepsilon \\ \\ \int\limits_{-\infty}^{\infty} \left[u_\varepsilon(x)^2 + v_\varepsilon(x)^2\right] dx = 1, \end{cases} \tag{14}$$

so, we have the system of integro-differential equations. To solve the system numerically we restrict the coordinate x to a compact interval $x = \frac{a}{1-|a|}$, so $a \in [-1, 1]$. The system is solved iteratively using Newton-Raphson method, based on 6th order central finite difference scheme. The resulting system of linear algebraic equations is solved with direct PARDISO solver. The simulations use a grid size of $N = 1000$ nodes, selected runs were repeated with other values of N to check the stability of our results. The relative errors of calculations are lower than 10^{-8}. We use the solutions of the system of equations without backreaction as the initial approximation in massless case, which can be found using Wolfram. In massive case we use massless solutions for this purpose.

In accordance with the topological properties of the kink and the localization of the fermion the boundary conditions are as follows:

$$\begin{cases} \phi|_{a=-1} = -1, & \phi|_{a=1} = 1, \\ u_\varepsilon|_{a=-1} = 0, & u_\varepsilon|_{a=1} = 0, \\ v_\varepsilon|_{a=-1} = 0, & v_\varepsilon|_{a=1} = 0. \end{cases} \tag{15}$$

Note, that due to the translational mode of the kink, i.e. the presence of x_0 in (9), the condition for the uniqueness of the solution is violated. To avoid it we use a boundary condition at an intermediate point.

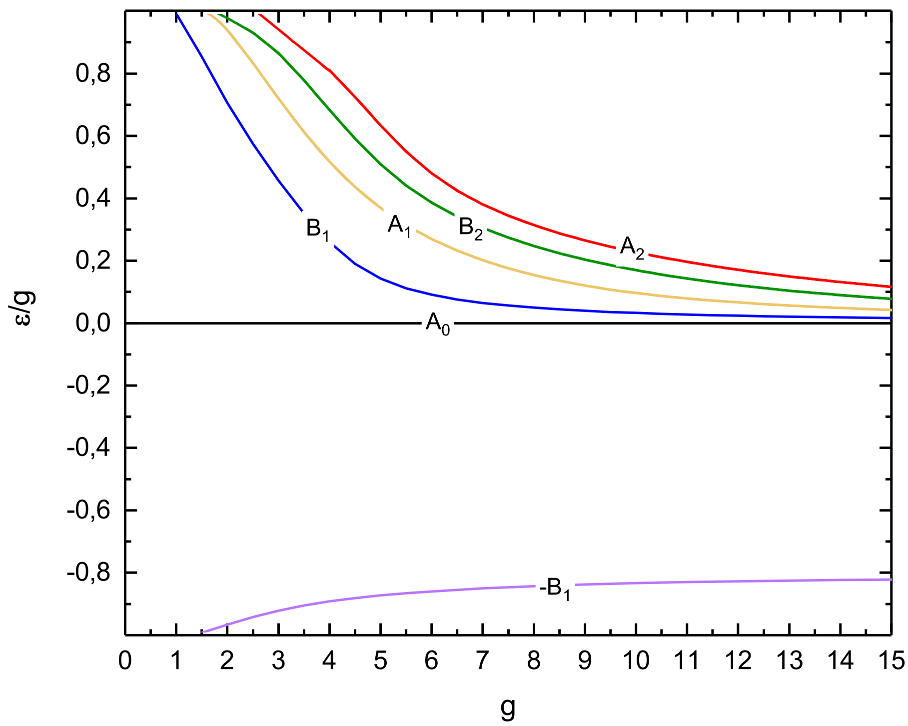

Figure 1: Normalized energy $\frac{\varepsilon}{g}$ of the localized fermionic states as a function of the fermion-kink coupling g for several fermion modes at $m = 0$.

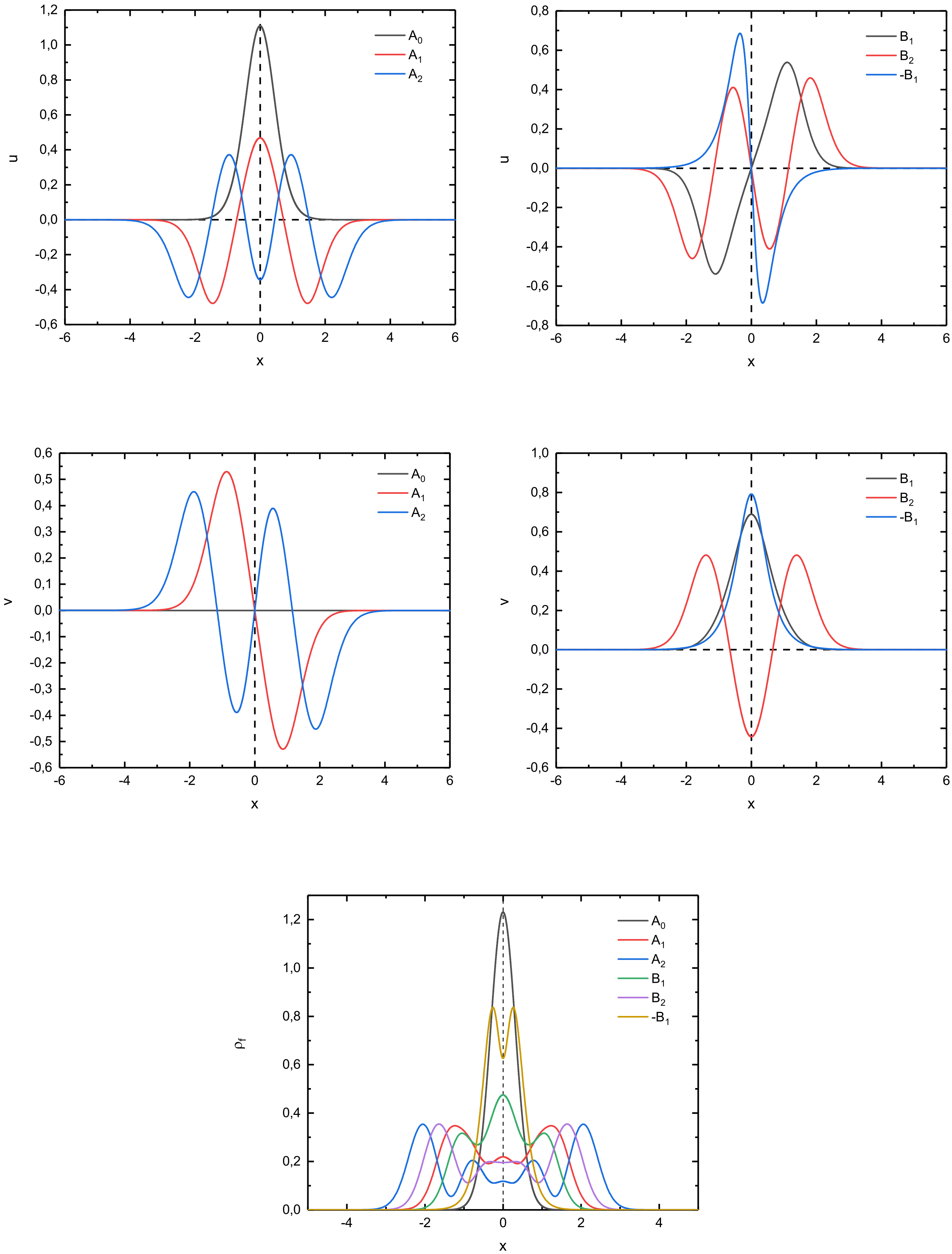

Figure 2: Fermionic field profile functions u (upper), v (middle) and a fermionic density ρ_f (lower) as functions of a coordinate x for several modes at $g = 5$, $m = 0$.

3.2 Massless fermions

First, consider the case of massless fermion, i.e. $m = 0$. In these section we use $a = 0$ as an intermediate point and the corresponding boundary condition is $\phi|_{a=0} = 0$. Note that in this case expansion of (11) near $a = 0$ gives us two types of boundary conditions for fermionic functions u_ε and v_ε:

$$\begin{cases} u'_\varepsilon|_{a=0} = 0, \\ v_\varepsilon|_{a=0} = 0, \end{cases} \qquad \text{or} \qquad \begin{cases} u_\varepsilon|_{a=0} = 0, \\ v'_\varepsilon|_{a=0} = 0. \end{cases} \tag{16}$$

Modes corresponding to the left boundary conditions will be called A_k-modes,

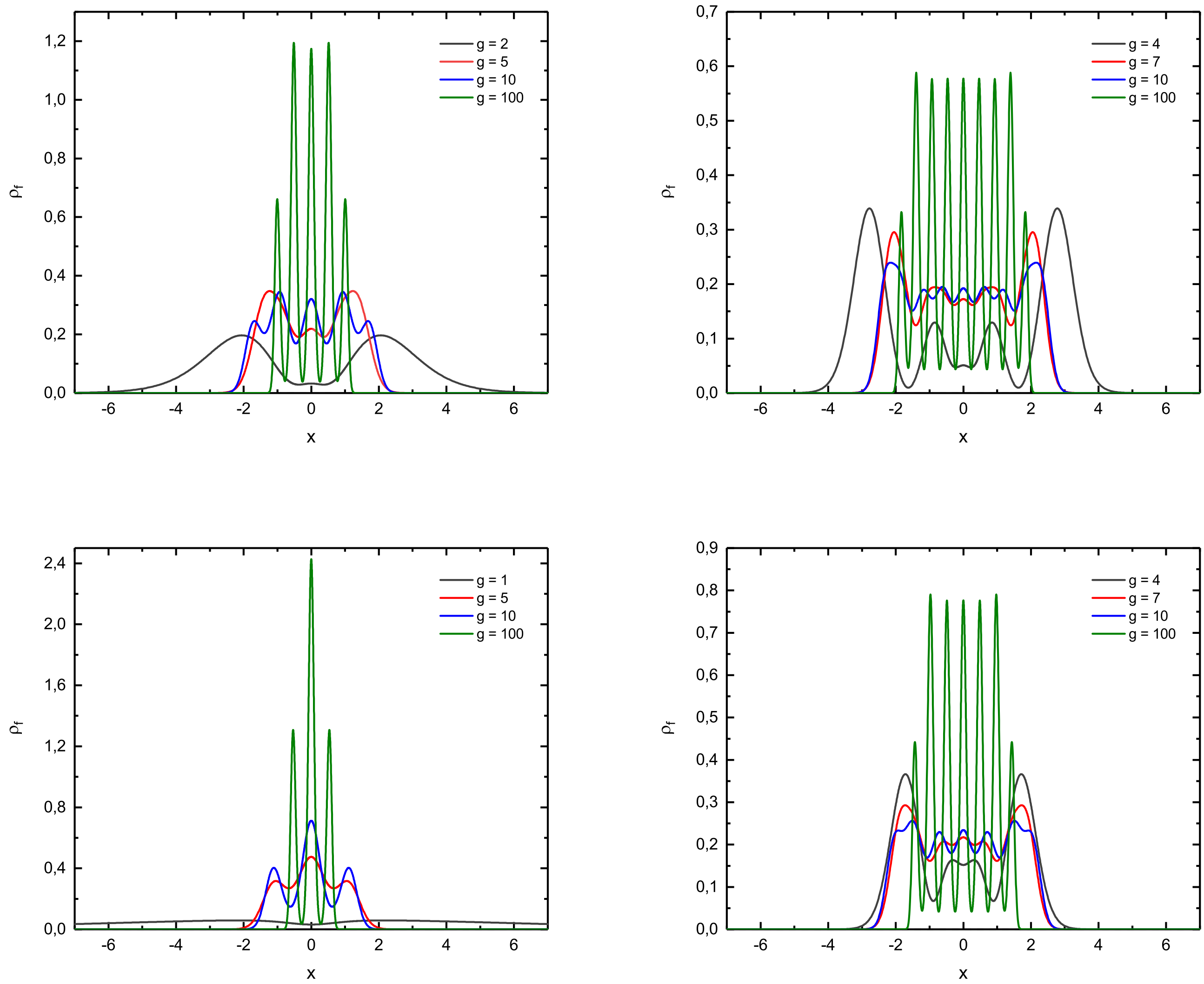

Figure 3: The fermionic densities ρ_f for modes A_1 (upper left), A_2 (upper right), B_1 (lower left) and B_2 (lower right) as functions of a coordinate x for several values of g at $m = 0$.

while modes corresponding to the left boundary conditions – B_k-modes. In these notations, the zero mode is A_0-mode.

The energy spectrum ε of fermions coupled to the undeformed kink was considered in several papers ([3] - [6]). In this approximation the profile function $\phi(x)$ of the kink is not affected by the coupling strength g. It corresponds to the case of a weak coupling limit of our numerical simulations. The pattern becomes different as we take into consideration the backreaction of the strongly coupled localized fermions. Indeed, for a finite value of the coupling strength g, the profile of the kink deforms as the fermion occupied an energy level, further, the energy levels move accordingly. In this sense, as the coupling g grows, we obtain an infinite tower of new solutions corresponding to the different modes (Fig. 1). It should be noted that accounting for backreaction is considered in [2], but with one wrong assumption.

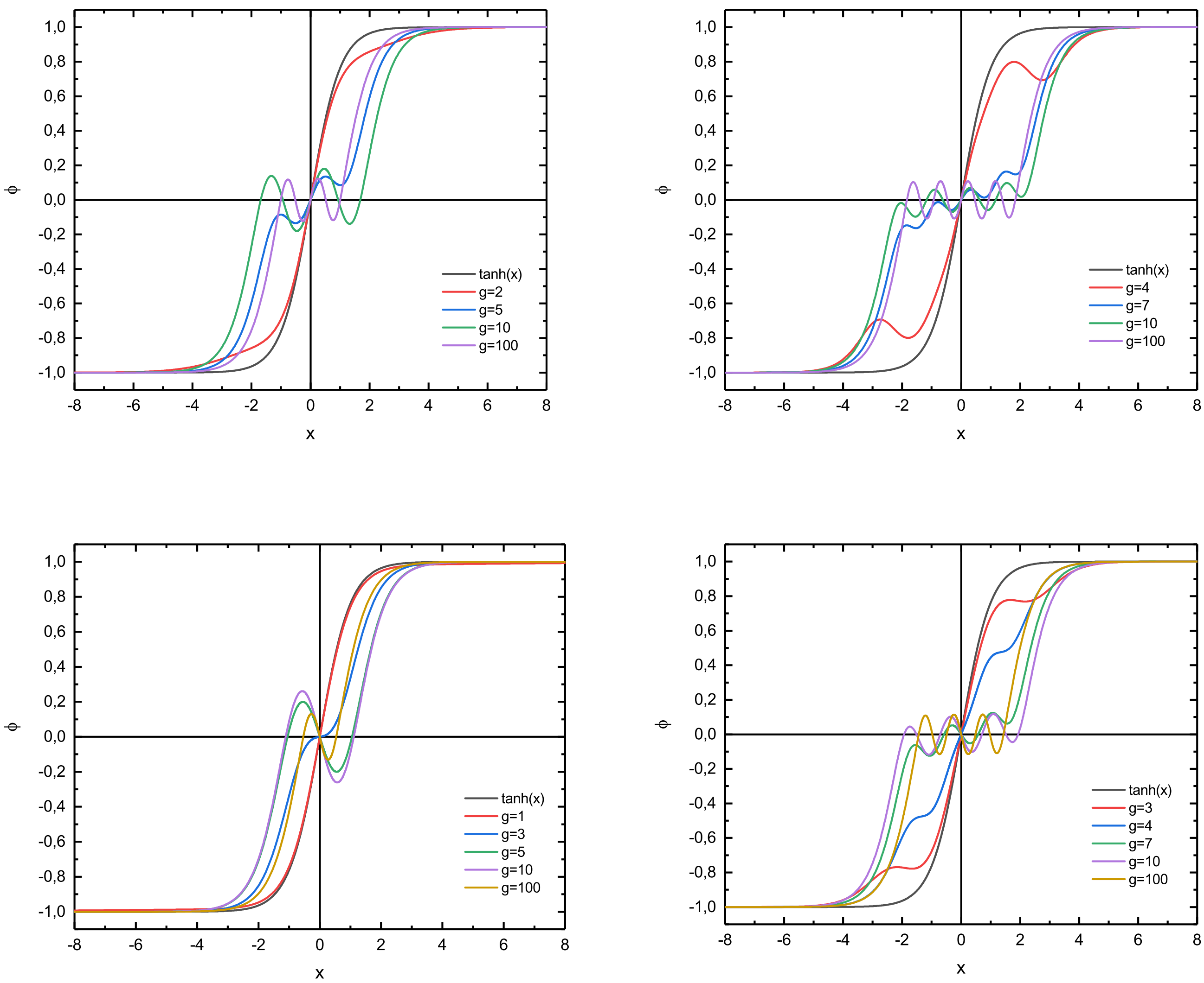

Figure 4: Kink profiles ϕ for modes A_1 (upper left), A_2 (upper right), B_1 (lower left) and B_2 (lower right) as functions of a coordinate x for several values of g at $m = 0$.

In order to exactly see the difference, in Fig. 2, we plot the components of the wave function u, v and the fermionic density ρ_f of the A_k, B_k. As can be seen, with an increase in the energy ε of the fermion (or with an increase in the number of the mode k) the fermion's spreading width increases and the fermion becomes less localized. But the localization of the fermion increases when the coupling constant grows up, which is confirmed by the Fig. 3.

Deformations of the kink profile for several modes are shown in Fig. 4. As stated, the zero mode does not distort the kink. Other modes (excited states) with an increase in the coupling constant more and more distort the kink profile. In this case, passing through a certain value of g, regions with a negative density of the topological charge appear. Note that depending on the mode, the number of such regions changes: the higher the mode, the more these regions. With strong coupling a region near zero with kink profile oscillations between positive and negative potential values appears. From a physical point of view, oscillations in the density of the topological charge of a coupled kink-fermion system can be interpreted as the birth of strongly coupled kink-antikink pairs. In this case, the topological charge of the system remains unchanged.

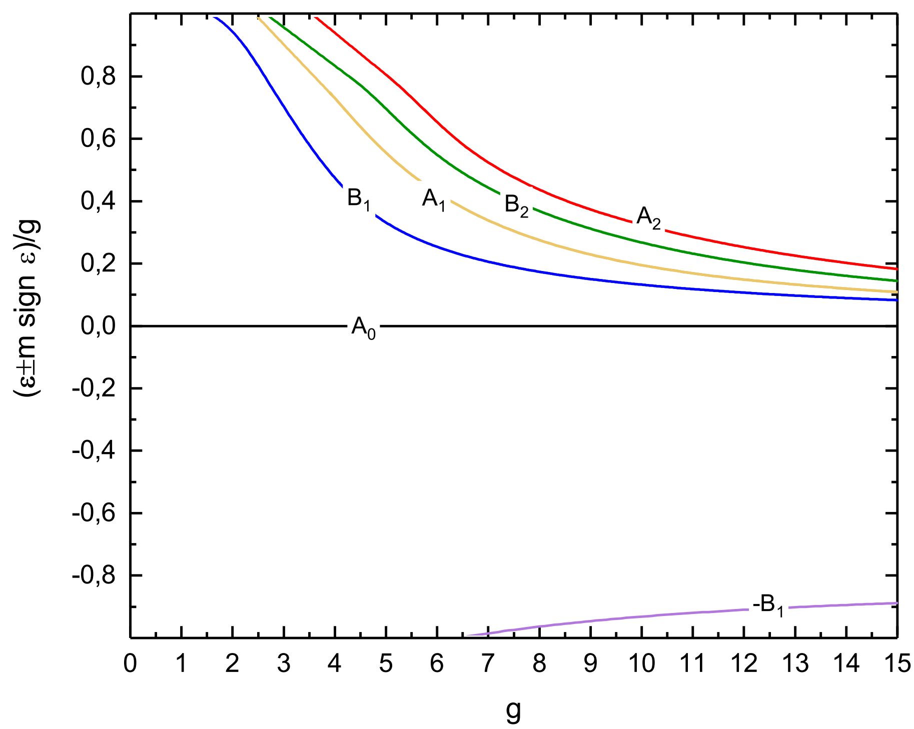

Figure 5: Normalized energy $\frac{\varepsilon + m\,\mathrm{sign}\,\varepsilon}{g}$ of the localized fermionic states as a function of the fermion-kink coupling g for several fermion modes at $m = 1$.

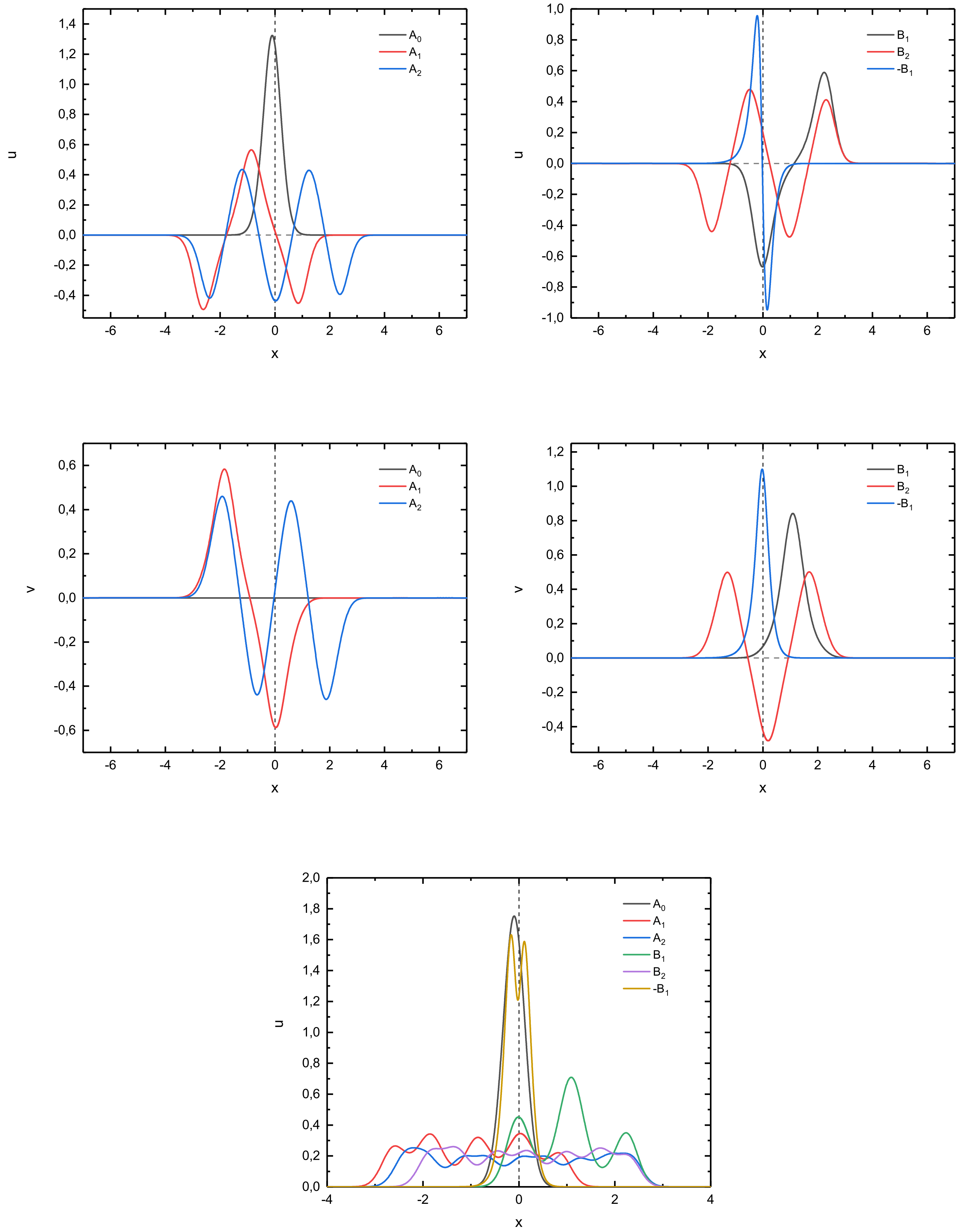

Figure 6: Fermionic field profile functions u (upper), v (middle) and a fermionic density ρ_f (lower) as functions of a coordinate x for several modes at $g = 10$, $m = 1$.

3.3 Massive fermions

Due to some minor features appearing in the massive case we leave unchanged the intermediate boundary condition only for mode $-B_1$. We use condition $\phi|_{a=0.7} = 0$ for modes $B_{1,2}$ and A_2 and $\phi|_{a=0.6} = 0.5$ for the A_1-mode.

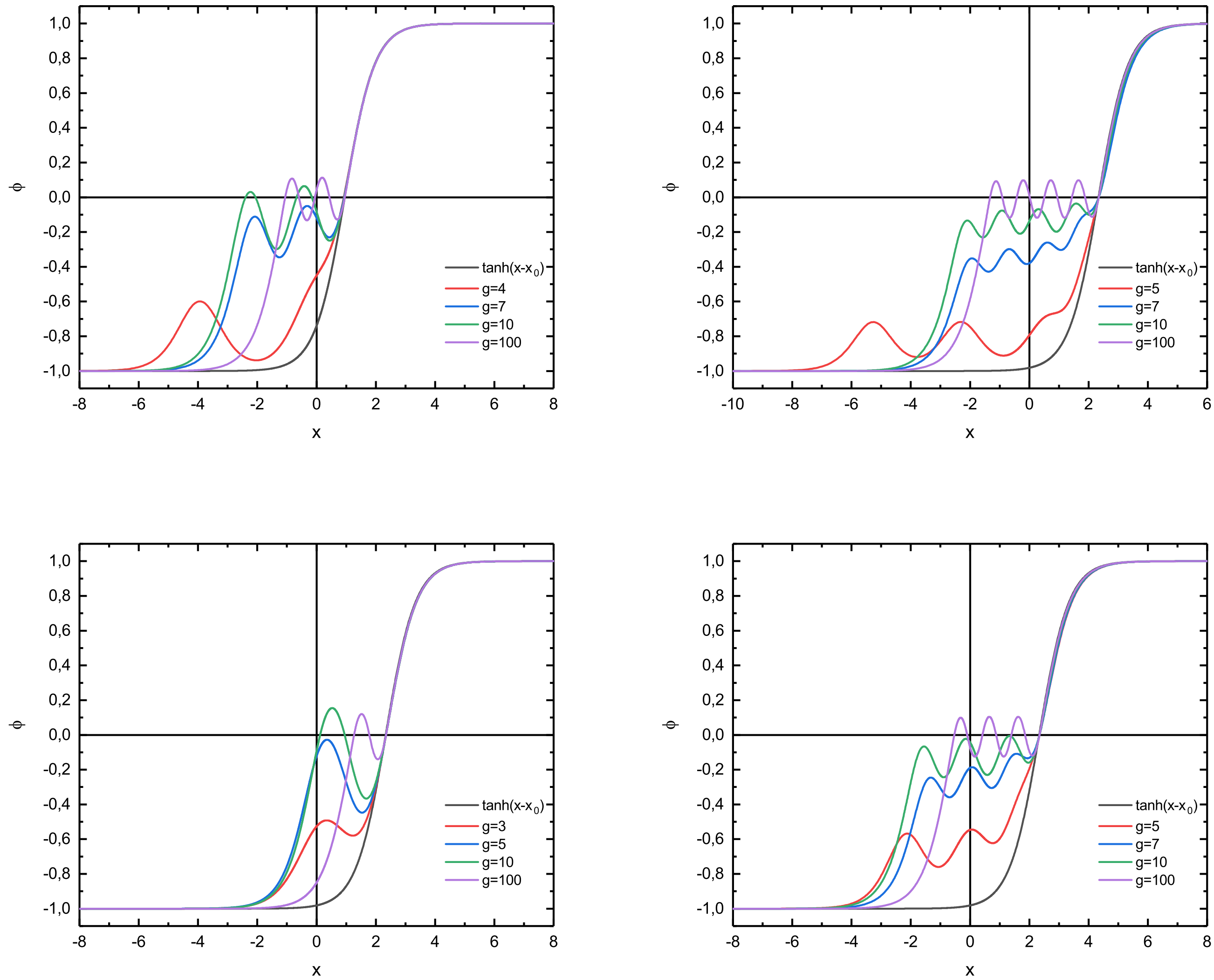

Figure 7: Kink profiles ϕ for modes A_1 (upper left), A_2 (upper right), B_1 (lower left) and B_2 (lower right) as functions of a coordinate x for several values of g at $m = 1$.

The results in the massive case differ only slightly from the massless one. Thus, as you can see in Fig. 5, excited modes appear at higher values of g in contrast to the massless case (Fig. 1). Note that in the presence of a fermion mass, the symmetry of fermionic functions is violated even in the absence of backreaction [6]. The situation is the same when taking into account the backreaction (Fig. 6). Moreover, due to the backreaction from the fermion, the kink profile loses its symmetry (strictly speaking, antisymmetry), as shown in Fig. 7. It is notable fact that in the massive case regions with a negative topological charge are wider than ones in the massless case.

Next, we consider the dependence of solutions on the value of the fermion mass m for fixed g (Fig. 8). The energy of the localized fermionic states is restricted as $|\varepsilon| < |g - m|$. Excitation modes are delocalizing at some critical values of the fermion mass m. On the other hand, decrease of the coupling constant g also leads to delocalization of the fermionic modes, only the zero mode A_0 remains as $g \ll 1$, see Fig. 1, 5.

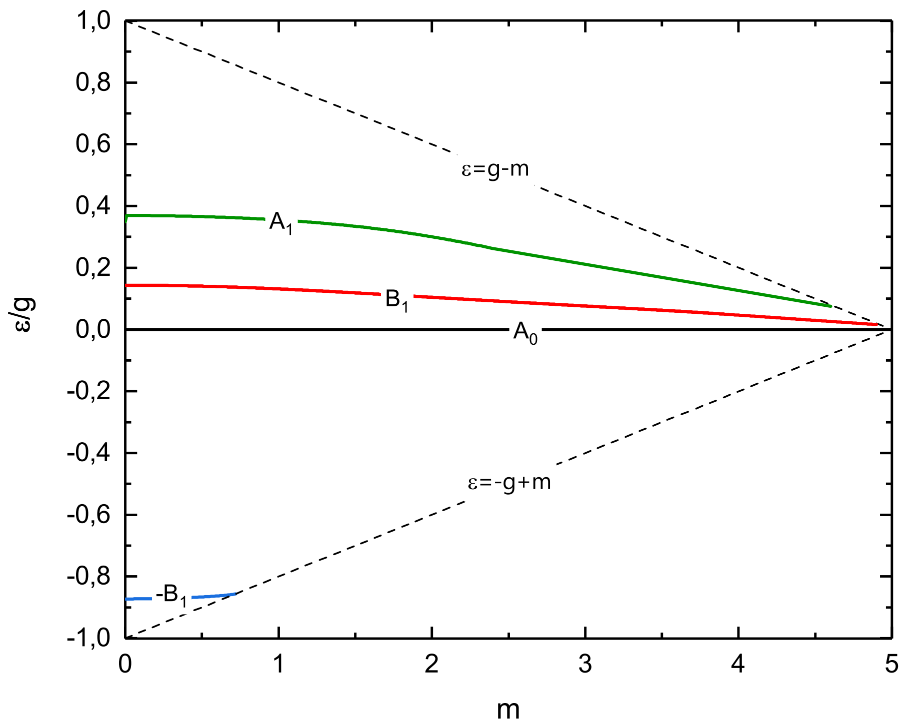

Figure 8: Normalized energy $\frac{\varepsilon}{g}$ of the localized fermionic states as a function of the fermion mass m for several modes at $g = 5$.

4 Conclusions

The objective of this work is to investigate the effects of backreaction of the fermions coupled to the kink in the ϕ^4-model. We found that there are two different types of the fermionic modes, localized on the kink, in particular there is only one zero mode. Other modes, which are linked to the positive and negative continuum, do not cross zero. Decrease of the coupling constant leads to delocalization of the fermionic modes, only the zero mode remains in the weak coupling limit. Considering the strong coupling limit we found that the coupling to the fermionic modes strongly deforms the kink, in particular we observe production of tightly bounded kink-anti-kink pairs.

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