JOINT INSTITUTE FOR NUCLEAR RESEARCH

Bogoliubov Laboratory of Theoretical Physics

FINAL REPORT ON THE SUMMER STUDENT PROGRAM

THERMODYNAMICS OF THE REAL SCALAR FIELD IN A FINITE VOLUME

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Participation period:
July 01 – August 11

Dubna, 2018

ABSTRACT

The Euler-Lagrange and Klein-Gordon Equations for the real and charged scalar field were obtained. The Energy-Momentum Tensor for the real and charged scalar field were also derived. The Hamiltonian density was obtained through the Lagrange method and from the Energy-Momentum Tensor. The current density was obtained. The solutions of the Klein-Gordon Equation were verified. The momentum and energy for the real scalar field were obtained. The Noether Charge and Current was calculated. In order to use the method of the canonical quantization, a profound study of the one-dimensional Harmonic Oscillator was made and the results were generalize to a system of harmonic oscillators. This knowledge was then used to quantize the Klein-Gordon field. The partition function of the real scalar field, considering the chemical potential equal to zero, was then obtained and the main thermodynamic quantities, such as thermodynamic potential, average of the number of particles, average energy, average momentum and pressure were then calculated. The partition function in the configuration space in one spatial dimension was obtained once again through the path integral method and, using the discrete Fourier transform, the partition function in the momentum space was obtained and the generalization to the three spatial dimension was made. The main thermodynamic quantities on the lattice were derived and, taking the continuum limit, the energy and pressure were obtained and compared to the ones obtained through the method of the canonical quantization.

I. INTRODUCTION

There are many methods of field quantization, such as the canonical method, and the path integral method, among others. The thermodynamic properties of the QCD phase diagram are described by both the canonical quantization formalism (second quantization) and the path-integral formalism.

The main aim of the project concluded at the Summer Student Program at JINR, which the present report describes, was to obtain the partition function for the real scalar field in a finite volume by the canonical quantization method and by the path integral method to then derive some main thermodynamic quantities and compare the results obtained through the two methods.

The procedure used for the quantization of fields leads us to the concept of quantized fields. The quantized wave field is a fundamental physical concept, within the framework of which the properties of elementary particles and their interactions are formulated.

The sections II and III are devoted to the development of the canonical quantization method. In section II is described a study of the classical theory of free fields where not only the the real scalar field was considered, but also the charged scalar field.

If we consider a continuous system as a discrete mechanical system with an infinite number of degrees of freedom, or, more precisely, an infinite number of field oscillators, we are enable to use the apparatus of classical mechanics in studying the field. That is just what was made at section III. First we made a study of the harmonic oscillator and then used this knowledge to quantize the Klein-Gordon Field and all the main results obtained in section II were rewritten in terms of operators. With that, in section IV we were able to derive the partition function using the method of the canonical quantization.

In section V the analytical derivations given in the paper [6] for the free real scalar field in the framework of the path integral method were reproduced. The partition function was obtained in the configuration and momentum space and some thermodynamic quantities were obtained and the results compared to the ones obtained at section IV.

The main conclusions are summarized in section VI.

II. CLASSICAL THEORY OF FREE FIELDS

We begin by considering the classical theory of relativistic wave fields.

A. Derivation of the Euler-Lagrange Equations

In this subsection we deduce the equations of motion, also called Euler-Lagrange Equations. First we deduce them for the neutral scalar field and after that for the charged scalar field.

Let us consider a system consisting only of the neutral scalar field $\phi(x)$ with mass m. In this case the Lagrangian density $\mathcal{L}$ is as given by Eq.(4.1) from Ref.[5]

$$\mathcal{L}(\phi, \partial_\mu \phi) = \frac{1}{2}(\partial^\mu \phi \partial_\mu \phi - m^2 \phi^2), \tag{1}$$

where we considered $c = \hbar = 1$.

The corresponding classical action S is the integral over space-time:

$$S = \int dt L = \int d^4 x \mathcal{L}\left(\phi(x), \partial_\mu \phi(x)\right), \tag{2}$$

as seen in Eq. (1.4) from Ref.[1]. And we invoke the Variational Principle, which establishes that the action must be invariant, that means:

$$\delta S = 0. \tag{3}$$

It leads us to:

$$\delta S = \delta \int \mathcal{L}\left(\phi(x), \partial_\mu \phi(x)\right) d^4 x = 0. \tag{4}$$

Now we need to calculate the variation of the Lagrangian density. In order to do so, let us first consider a function of two independent variables $f(x_1, x_2)$. The variation of this function is:

$$\delta f(x_1, x_2) = f(x_1 + \Delta x_1, x_2 + \Delta x_2) - f(x_1, x_2). \tag{5}$$

The Taylor expansion of $f(x_1 + \Delta x_1, x_2 + \Delta x_2)$ can be written as:

$$\begin{aligned} \delta f(x_1 + \Delta x_1, x_2 + \Delta x_2) = f(x_1, x_2) + \frac{1}{1!}\frac{\partial f(x_1, x_2)}{\partial x_1}\Delta x_1 + \frac{1}{1!}\frac{\partial f(x_1, x_2)}{\partial x_2}\Delta x_2 + \\ + \frac{1}{2!}\frac{\partial^2 f(x_1, x_2)}{\partial x_1^2}(\Delta x_1)^2 + \frac{1}{2!}\frac{\partial^2 f(x_1, x_2)}{\partial x_2^2}(\Delta x_2)^2 + \frac{2}{2!}\frac{\partial^2 f(x_1, x_2)}{\partial x_1 \partial x_2}\Delta x_1 \Delta x_2 + ... \end{aligned} \tag{6}$$

Substituting eq. (6) into eq. (5) and neglecting the terms of second order, we have:

$$\delta f(x_1, x_2) = \frac{\partial f(x_1, x_2)}{\partial x_1}\Delta x_1 + \frac{\partial f(x_1, x_2)}{\partial x_2}\Delta x_2 \tag{7}$$

Looking at our Lagrangian density we can see that it is also a function of two independent variables $\mathcal{L} = \mathcal{L}(\phi, \partial_\mu \phi)$. Therefore we just follow the recipe above and we have:

$$\delta \mathcal{L}(\phi, \partial_\mu \phi) = \frac{\partial \mathcal{L}(\phi, \partial_\mu \phi)}{\partial \phi}\delta \phi + \frac{\partial \mathcal{L}(\phi, \partial_\mu \phi)}{\partial(\partial_\mu \phi)}\delta(\partial_\mu \phi). \tag{8}$$

According to eq. (4):

$$\delta S = \int d^4 x \left(\frac{\partial \mathcal{L}(\phi, \partial_\mu \phi)}{\partial \phi}\delta \phi + \frac{\partial \mathcal{L}(\phi, \partial_\mu \phi)}{\partial(\partial_\mu \phi)}\delta(\partial_\mu \phi) \right) = 0. \tag{9}$$

Let us take a look at the second term of this equation. Integrating by parts, we have:

$$\int d^4 x \left(\frac{\partial \mathcal{L}(\phi, \partial_\mu \phi)}{\partial(\partial_\mu \phi)} \right) \delta(\partial_\mu \phi) = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta \phi - \int d^4 x \left[\left(\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \delta \phi \right]. \tag{10}$$

We assume that the variations of the field functions $\delta\phi$ vanish at the surface of the four-volume over which the integral is taken. Therefore we obtain:

$$\delta S = \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \right] \delta\phi = 0. \tag{11}$$

As this equations are valid for arbitrary variations of the scalar field $\delta\phi$, we finally have the Euler-Lagrange Equation for the real scalar field:

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = 0. \tag{12}$$

Let us do the same for the charged scalar field. In this case, as can be seen on Eq.(4.54) from Ref.[5], we have the following Lagrangian density:

$$\mathcal{L}(\phi, \phi^*, \partial_\mu \phi, \partial_\mu \phi^*) = (\partial^\mu \phi^* \partial_\mu \phi - m^2 \phi^* \phi). \tag{13}$$

The variation of $\mathcal{L}$ can be written as:

$$\delta\mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta(\partial_\mu \phi) + \frac{\partial \mathcal{L}}{\partial \phi *}\delta\phi * + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi *)}\delta(\partial_\mu \phi *). \tag{14}$$

Then the Variation Principle gives us:

$$\delta S = \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta(\partial_\mu \phi) + \frac{\partial \mathcal{L}}{\partial \phi *}\delta\phi^* + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)}\delta(\partial_\mu \phi *) \right] = 0. \tag{15}$$

Integrating by parts again the second and last term, we obtain

$$\delta S = \int \left\{ \delta\phi \left[\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu)\phi} \right) \right] + \delta\phi^* \left[\frac{\partial \mathcal{L}}{\partial \phi^*} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} \right) \right] \right\} d^4x = 0. \tag{16}$$

As ϕ and ϕ^* are independent fields and these equations are valid for arbitrary variations of $\delta\phi$ and $\delta\phi^*$, we have two Euler-Lagrange Equations. For ϕ:

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = 0 \tag{17}$$

which is the same as that obtained earlier. And for ϕ^*:

$$\frac{\partial \mathcal{L}}{\partial \phi^*} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} \right) = 0. \tag{18}$$

B. Derivation of the Klein-Gordon Equation

Now that we have the Euler-Lagrange Equations we can also derive the Klein-Gordon Equations. For the real scalar field we just introduce the Lagrangian density into eq. (12)

$$\frac{\partial \mathcal{L}}{\partial \phi} = -m^2 \phi \tag{19}$$

and

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = \partial_\mu \partial^\mu \phi. \tag{20}$$

Identifying $\partial_\mu \partial^\mu$ as the d'Alembert operator $\Box$ we have

$$(\Box + m^2)\phi(x) = 0 \tag{21}$$

which is called the Klein-Gordon Equation for the real scalar field [5].

Now we will do the same for the charged scalar field. We need the Lagrangian density given by eq. (13) and the Euler-Lagrange equations given by eqs.(17) and (18). Let's start with the eq. (17). We have

$$\frac{\partial \mathcal{L}}{\partial \phi} = -m^2 \phi^* \tag{22}$$

and

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = \partial_\mu \partial^\mu \phi^*. \tag{23}$$

Substituting eqs. (22) and (23) into eq. (17), we obtain the Klein-Gordon equation

$$(\Box + m^2)\phi^*(x) = 0. \tag{24}$$

Equation (18) gives us

$$\frac{\partial \mathcal{L}}{\partial \phi^*} = -m^2 \phi \tag{25}$$

and

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} \right) = \partial_\mu \partial^\mu \phi. \tag{26}$$

Then we have the Klein-Gordon equation as

$$(\Box + m^2)\phi(x) = 0. \tag{27}$$

Note that this equation is equal to eq. (21).

C. Derivation of the Energy-Momentum Tensor

To calculate the energy momentum tensor $T_{\mu\nu}$ of the neutral scalar field we consider again the Lagrangian density as in eq. (1). But now we derive it in relation to the coordinates x^μ. As $\mathcal{L}$ does not explicitly depend on the coordinates, we have to use the chain rule of differentiation:

$$\frac{\partial \mathcal{L}}{\partial x^\mu} = \frac{\partial \mathcal{L}}{\partial \phi} \frac{\partial \phi}{\partial x^\mu} + \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} \frac{\partial(\partial_\nu \phi)}{\partial x^\mu}. \tag{28}$$

Remembering the Euler-Lagrange Equation for the scalar field as in eq. (12), we have:

$$\frac{\partial \mathcal{L}}{\partial \phi} = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right). \tag{29}$$

Substituting this in the equation above, we are left with:

$$\begin{aligned}\partial_\mu \mathcal{L} &= \partial_\nu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)}\right)\partial_\mu \phi + \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)}\partial_\mu(\partial_\nu \phi) \\ &= \partial_\nu \left[\left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)}\right)\partial_\mu \phi\right],\end{aligned} \tag{30}$$

where we used $\partial_\mu = \dfrac{\partial}{\partial x^\mu}$. Continuing the calculation and using $\partial_\mu = g_\mu^\nu \partial_\nu$, we get

$$\partial_\nu \left[\left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)}\right)\partial_\mu \phi - g_\mu{}^\nu \mathcal{L}\right] = 0, \tag{31}$$

where we identify the energy momentum tensor with [3]

$$T^\nu{}_\mu = \left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)}\right)\partial_\mu \phi - g_\mu{}^\nu \mathcal{L}. \tag{32}$$

Then we have:

$$\partial_\nu T^\nu{}_\mu = 0. \tag{33}$$

Since $g_{\rho\nu} T^\nu{}_\mu = T_{\rho\nu}$, we now have:

$$T_{\rho\mu} = \left(\frac{\partial \mathcal{L}}{\partial(\partial^\rho \phi)}\right)\partial_\mu \phi - g_{\rho\nu} g_\mu{}^\nu \mathcal{L}. \tag{34}$$

Noting that $g_{\rho\nu} g_\mu{}^\nu = g_{\rho\mu}$ and making the index changes $\rho \to \mu$ and $\mu \to \nu$, we finally have the energy momentum tensor in the form

$$T_{\mu\nu} = \left(\frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi)}\right)\partial_\nu \phi - g_{\mu\nu} \mathcal{L}, \tag{35}$$

which is in accordance to Eq. (2.58) from Ref.[5]. If we replace our Lagrangian density in the tensor, we will end up with:

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - g_{\mu\nu} \frac{1}{2}\left[\partial^\sigma \phi \partial_\sigma \phi - m^2 \phi^2\right]. \tag{36}$$

We can also deduce the energy-momentum tensor for charged scalar field. Remembering that in this case we have the Lagrangian density as eq. (13). In this case, the derivation in relation do x^μ is going to have two more terms, as follows:

$$\frac{\partial \mathcal{L}}{\partial x^\mu} = \frac{\partial \mathcal{L}}{\partial \phi}\frac{\partial \phi}{\partial x^\mu} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\frac{\partial(\partial_\mu \phi)}{\partial x^\mu} + \frac{\partial \mathcal{L}}{\partial \phi^*}\frac{\partial \phi^*}{\partial x^\mu} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)}\frac{\partial(\partial_\mu \phi^*)}{\partial x^\mu}. \tag{37}$$

From the Euler-Lagrange Equations for the charged field, as seen in eqs. (17) and (18), we have:

$$\frac{\partial \mathcal{L}}{\partial \phi} = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right), \tag{38}$$

$$\frac{\partial \mathcal{L}}{\partial \phi^*} = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)}\right). \tag{39}$$

Replacing it in the equation above, we obtain

$$\begin{aligned}\partial_\mu \mathcal{L} &= \partial_\nu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} \right) \partial_\mu \phi + \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} \partial_\mu(\partial_\nu \phi) + \partial_\nu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi^*)} \right) \partial_\mu \phi^* + \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi^*)} \partial_\mu(\partial_\nu \phi^*) \\ &= \partial_\nu \left[\left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} \right) \partial_\mu \phi + \left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi^*)} \right) \partial_\mu \phi^* \right]. \end{aligned} \tag{40}$$

Making use of the relation $\partial_\mu = g_\mu{}^\nu \partial_\nu$ we end up with

$$\partial_\nu \left[\left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} \right) \partial_\mu \phi + \left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi^*)} \right) \partial_\mu \phi^* - g_\mu{}^\nu \partial_\nu \mathcal{L} \right] = 0. \tag{41}$$

Identifying the energy-momentum tensor as:

$$T^\nu{}_\mu = \left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} \right) \partial_\mu \phi + \left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi^*)} \right) \partial_\mu \phi^* - g_\mu{}^\nu \mathcal{L} \tag{42}$$

and doing the same calculations we did for the scalar field, we end up with the final form of the energy-momentum tensor for the charged field:

$$T_{\mu\nu} = \left(\frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi)} \right) \partial_\nu \phi + \left(\frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi^*)} \right) \partial_\nu \phi^* - g_{\mu\nu} \mathcal{L}, \tag{43}$$

which is equal to eq.(3.33) from Ref.[1]. We can also substitute the Lagrangian density given by eq.(13) into this equation in order to calculate the energy-momentum tensor explicitly

$$T_{\mu\nu} = \partial_\mu \phi^* \partial_\nu \phi + \partial_\mu \phi \partial_\nu \phi^* - g_{\mu\nu}(\partial^\sigma \phi^* \partial_\sigma \phi - m^2 \phi^* \phi). \tag{44}$$

D. Two ways to derive the Hamiltonian density

First we will derive the Hamiltonian density for the real scalar field using the Legendre transform. In order to do so, let us first calculate the canonically conjugate field. Remembering that the Lagrangian density to be used here is the one given by eq. (1):

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial(\dot{\phi}(x))} = \dot{\phi}(x). \tag{45}$$

And now we can calculate the Hamiltonian density as follows [5]

$$\begin{aligned}\mathcal{H}(x) &= \pi(x)\dot{\phi}(x) - \mathcal{L}(x) \\ &= \partial_0 \phi \partial_0 \phi - \frac{1}{2}(\partial_0 \phi \partial_0 \phi - \partial_i \phi \partial_i \phi - m^2 \phi^2), \end{aligned} \tag{46}$$

where we used the fact that $\partial_\mu = \left(\frac{\partial}{\partial t}, \boldsymbol{\nabla} \right)$ and $\partial^\mu = \left(\frac{\partial}{\partial t}, -\boldsymbol{\nabla} \right)$. And we end up with

$$\mathcal{H}(x) = \frac{1}{2} \left(\pi^2(x) + (\boldsymbol{\nabla} \phi(x))^2 + m^2 \phi^2(x) \right). \tag{47}$$

This equation is equal to Eq.(4.4) from Ref.[5].

We can also calculate the Hamiltonian density for the charged scalar field. In this case we have to use the Lagrangian density given by eq. (13). The canonically conjugate fields are

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi}^*, \tag{48}$$

$$\pi^* = \frac{\partial \mathcal{L}}{\partial \dot{\phi}^*} = \dot{\phi}. \tag{49}$$

The Hamiltonian density in this case is

$$\begin{aligned}
\mathcal{H} &= \pi\dot{\phi} + \pi^*\dot{\phi}^* - \mathcal{L} \\
&= \pi\pi^* + \pi^*\pi - (\partial^\mu\phi^*\partial_\mu\phi - m^2\phi^*\phi) \\
&= \partial^0\phi\partial_0\phi^* + \partial^0\phi^*\partial_0\phi - (\partial^0\phi^*\partial_0\phi + \partial^i\phi^*\partial_i\phi - m^2\phi^*\phi) \\
&= \pi\pi^* - (-\boldsymbol{\nabla}\phi^*)(\boldsymbol{\nabla}\phi) + m^2\phi^*\phi
\end{aligned} \tag{50}$$

and we finally have

$$\mathcal{H}(x) = \pi(x)\pi^*(x) + \boldsymbol{\nabla}\phi^*(x)\boldsymbol{\nabla}\phi(x) + m^2\phi^*(x)\phi(x). \tag{51}$$

The other way to derive the Hamiltonian density is through the relation $\mathcal{H} = T_{00}$, as can be seen on Eq.(1.58) from [3]. So, if we take the energy-momentum tensor given by eq. (36), we obtain the Hamiltonian density for the real scalar field:

$$\begin{aligned}
T_{00} &= (\partial_0\phi)^2 - g_{00}\left[\frac{1}{2}(\partial^\sigma\phi\partial_\sigma\phi - m^2\phi^2)\right] \\
&= (\partial_0\phi)^2 - \frac{1}{2}(\partial^0\phi\partial_0\phi) - \frac{1}{2}(\partial^i\phi\partial_i\phi) + \frac{1}{2}(m^2\phi^2) \\
&= \frac{1}{2}\left[\dot{\phi}^2 + (\boldsymbol{\nabla}\phi)^2 + m^2\phi^2\right].
\end{aligned} \tag{52}$$

Remembering eq. (45), we finally have the Hamiltonian density

$$\mathcal{H}(x) = \frac{1}{2}\left(\pi^2(x) + (\boldsymbol{\nabla}\phi(x))^2 + m^2\phi^2(x)\right). \tag{53}$$

We can observe that this equation is equal to eq. (47).

If we take the energy-momentum tensor given by eq. (34), we find the Hamiltonian density for the charged scalar field

$$\begin{aligned}
T_{00} &= (\partial_0\phi^*\partial_0\phi + (\partial_0\phi\partial_0\phi^* - g_{00}\left[\partial^\sigma\phi^*\partial_\sigma\phi - m^2\phi^*\phi\right] \\
&= \partial_0\phi\partial_0\phi^* + \partial_i\phi^*\partial_i\phi + m^2\phi^*\phi \\
&= \dot{\phi}\dot{\phi}^* + (\boldsymbol{\nabla}\phi^*)(\boldsymbol{\nabla}\phi) + m^2\phi^*\phi.
\end{aligned} \tag{54}$$

Remembering eqs. (48) and (49), we end up with

$$\mathcal{H}(x) = \pi(x)\pi^*(x) + \boldsymbol{\nabla}\phi^*(x)\boldsymbol{\nabla}\phi(x) + m^2\phi^*(x)\phi(x). \tag{55}$$

We can observe that this equation is equal to eq. (51).

E. Current Density

In this section we derive the current density. For that, we need the Lagrangian density given by eq. (13), take it's variation as in eq. (14), and insert eqs. (38) and (39)

$$\begin{aligned}\delta\mathcal{L} &= \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\phi)}\right)\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta(\partial_\mu\phi) + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)}\right)\delta\phi^* + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)}\delta(\partial_\mu\phi^*)\\ &= \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)}\delta\phi^*\right] = 0,\end{aligned} \tag{56}$$

where we used

$$\delta(\partial_\mu\phi) = \partial_\mu(\phi+\delta\phi) - \partial_\mu\phi = \partial_\mu(\delta\phi). \tag{57}$$

Now, if we consider a small transformation of the field $\phi \to e^{ie\alpha}\phi$, we have the variation of this field being $\delta\phi = ie\alpha\phi$. So, substituting it in the equation above

$$\delta\mathcal{L} = -ie\alpha\partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)}\phi^* - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\phi\right] = 0. \tag{58}$$

Since [5]

$$\partial_\mu j^\mu = 0, \tag{59}$$

we can define the current density as

$$j^\mu = ie\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)}\phi^* - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\phi\right]. \tag{60}$$

This equation is in accordance with Eq.(2.28) from Ref.[1] if $e = 1$. Making the derivations of $\mathcal{L}$, we finally have the explicit form

$$j^\mu = ie(\phi\partial^\mu\phi^* - \phi^*\partial^\mu\phi). \tag{61}$$

We can observe that, if $\phi^* = \phi$, that is, if we have only a real scalar field, meaning that we have no charged field, the current density is going to be equal to zero.

F. Solutions of Klein-Gordon Equation

Considering the Klein-Gordon Equation given by eq. (21), let us see if the two equations below are solutions of this equation

$$\phi_1(x) = \frac{e^{-ipx}}{\sqrt{2EV}} \tag{62}$$

and

$$\phi_2(x) = \frac{e^{ipx}}{\sqrt{2EV}}, \tag{63}$$

where $px = p^\mu x_\mu = Et - \boldsymbol{pr}$, with $\boldsymbol{pr} = p_x x + p_y y + p_z z$. Starting with ϕ_1

$$(\partial_\mu \partial^\mu + m^2)\phi_1 = \left(\left(\frac{\partial}{\partial t}\right)^2 - \boldsymbol{\nabla}^2 + m^2\right)\frac{e^{-i(Et-\boldsymbol{pr})}}{\sqrt{2EV}}$$
$$= [(-iE)^2 + (i\boldsymbol{p})^2 + m^2]\frac{e^{-i(Et-\boldsymbol{pr})}}{\sqrt{2EV}}$$
$$= [-E^2 + \boldsymbol{p}^2 + m^2]\frac{e^{-i(Et-\boldsymbol{pr})}}{\sqrt{2EV}} = 0. \tag{64}$$

So we conclude that $-E^2 + \boldsymbol{p}^2 + m^2 = 0$ and we have the relativistic dispersion relation [5]

$$E = \sqrt{\boldsymbol{p}^2 + m^2}. \tag{65}$$

Here we are considering only the positive solution because it is the one that corresponds to the free particles.

For ϕ_2 the calculations are similar

$$(\partial_\mu \partial^\mu + m^2)\phi_2 = \left(\left(\frac{\partial}{\partial t}\right)^2 - \boldsymbol{\nabla}^2 + m^2\right)\frac{e^{i(Et-\boldsymbol{pr})}}{\sqrt{2EV}}$$
$$= [(iE)^2 + (-i\boldsymbol{p})^2 + m^2]\frac{e^{i(Et-\boldsymbol{pr})}}{\sqrt{2EV}}$$
$$= [-E^2 + \boldsymbol{p}^2 + m^2]\frac{e^{-i(Et-\boldsymbol{pr})}}{\sqrt{2EV}} = 0. \tag{66}$$

Resulting the same relativistic dispersion relation as given by eq. (65).

So, if the two equations are solutions of the Klein-Gordon Equation, a linear combinations of the two is also going to be a solution:

$$\phi(x) = \sum_{\boldsymbol{p}} \frac{1}{\sqrt{2EV}}(a_{\boldsymbol{p}} e^{-ipx} + b_{\boldsymbol{p}} e^{ipx}), \tag{67}$$

where $a_{\boldsymbol{p}}$ and $b_{\boldsymbol{p}}$ are constants.

As we are considering of a real scalar field, we have to impose $\phi^* = \phi$, which also gives us $b_{\boldsymbol{p}} = a^*_{\boldsymbol{p}}$ and eq.(67) can be written as

$$\phi(x) = \sum_{\boldsymbol{p}} \frac{1}{\sqrt{2EV}}(a_{\boldsymbol{p}} e^{-ipx} + a^*_{\boldsymbol{p}} e^{ipx}). \tag{68}$$

We can also write eq. (68) in terms of the wave vector $\boldsymbol{k}$. In order to do so, let's remember the relation [5]

$$p_\alpha = \frac{2\pi}{L} k_\alpha, \tag{69}$$

where we used the natural units so that $\hbar = c = 1$. Also $k_\alpha \in \mathbb{Z}$ and $\alpha = 1, 2, 3 = x, y, z$ and L is the size of one side of a cubic box in which the particle is confined so that $L = V^{1/3}$, where V is the volume of this box.

Substituting this in eq. (68), we have

$$\phi(x) = \sum_{\boldsymbol{k}=-\infty}^{\infty} \frac{1}{\sqrt{2EV}}[a_{\boldsymbol{k}} e^{-i(Et-\frac{2\pi}{L}\boldsymbol{kr})} + a^*_{\boldsymbol{k}} e^{i(Et-\frac{2\pi}{L}\boldsymbol{kr})}] \tag{70}$$

and the relativistic dispersion relation is now given by:

$$E = \sqrt{\left(\frac{2\pi}{L}\right)^2 (k_x^2 + k_y^2 + k_z^2) + m^2} = \sqrt{\left(\frac{2\pi \boldsymbol{k}}{L}\right)^2 + m^2}. \tag{71}$$

G. Calculation of the Momentum

We have that the momentum **P**, as can be seen by Eq.(4.47) from Ref.[5], is given by

$$\boldsymbol{P} = -\int d^3x T_{0i}, \tag{72}$$

where $d^3x = dxdydz$ and the minus sign came from the fact that, for any covariant four-vector we have $P_\mu = (P_0, -P_i) = (P_0, -\boldsymbol{P})$. Remembering the energy-momentum tensor given by eq. (36), we can calculate T_{0i} as

$$\begin{aligned} T_{0i} &= \partial_0\phi\partial_i\phi - g_{0i}\left[\frac{1}{2}(\partial^\sigma\phi\partial_\sigma\phi - m^2\phi^2)\right] \\ &= \frac{\partial\phi}{\partial t}\boldsymbol{\nabla}\phi. \end{aligned} \tag{73}$$

Using the function ϕ given by eq. (68) and taking the temporal and spatial derivations, we have

$$\begin{aligned} T_{0i} = \sum_{\boldsymbol{p'}}\sum_{\boldsymbol{p}} \frac{-E'\boldsymbol{p}}{\sqrt{4E'EV^2}}\Big(&- a_{\boldsymbol{p'}}a_{\boldsymbol{p}}e^{-i[t(E'+E)-\boldsymbol{r}(\boldsymbol{p'}+\boldsymbol{p})]} - a^*_{\boldsymbol{p'}}a^*_{\boldsymbol{p}}e^{i[t(E'+E)-\boldsymbol{r}(\boldsymbol{p'}+\boldsymbol{p})]} \\ &+ a_{\boldsymbol{p'}}a^*_{\boldsymbol{p}}e^{-i[t(E'-E)-\boldsymbol{r}(\boldsymbol{p'}-\boldsymbol{p})]} + a^*_{\boldsymbol{p'}}a_{\boldsymbol{p}}e^{i[t(E'-E)-\boldsymbol{r}(\boldsymbol{p'}-\boldsymbol{p})]}\Big). \end{aligned} \tag{74}$$

Let us take the first term above and do its integration

$$\begin{aligned} \int d^3x\left(a_{\boldsymbol{p'}}a_{\boldsymbol{p}}e^{-i[t(E'+E)-\boldsymbol{r}(\boldsymbol{p'}+\boldsymbol{p})]}\right) &= a_{\boldsymbol{p'}}a_{\boldsymbol{p}}e^{-it(E'+E)}\int d^3x e^{i\boldsymbol{r}(\boldsymbol{p'}+\boldsymbol{p})} \\ &= a_{\boldsymbol{p'}}a_{\boldsymbol{p}}e^{-it(E'+E)}\int_0^L dx e^{\frac{2\pi i}{L}x(k'_x+k_x)}\int_0^L dy e^{\frac{2\pi i}{L}y(k'_y+k_y)}\int_0^L dz e^{\frac{2\pi i}{L}z(k'_z+k_z)} \\ &= a_{\boldsymbol{p'}}a_{\boldsymbol{p}}e^{-it(E'+E)}\left(\frac{L}{2\pi i}\right)^3\left[\frac{(e^{2\pi i(k'_x+k_x)}-1)}{k'_x+k_x}\frac{(e^{2\pi i(k'_y+k_y)}-1)}{k'_y+k_y}\frac{(e^{2\pi i(k'_z+k_z)}-1)}{k'_z+k_z}\right]. \end{aligned} \tag{75}$$

Now we have two possible solutions: the case when $\boldsymbol{k'}+\boldsymbol{k}=0$ and the case when $\boldsymbol{k'}+\boldsymbol{k}\neq 0$. Let us take a look at the two cases. Analyzing only the part with dependence on x, taking the limit $k'_x + k_x \to 0$, we can make a Taylor expansion of the exponential

$$\frac{(e^{2\pi i(k'_x+k_x)}-1)}{k'_x+k_x} = \frac{1+2\pi i(k'_x+k_x)+\frac{1}{2}(2\pi i)^2(k'_x+k_x)^2+...-1}{(k'_x+k_x)} = 2\pi i. \tag{76}$$

In the case when $\boldsymbol{k'}+\boldsymbol{k}\neq 0$ we can use the Euler's formula

$$\frac{(e^{2\pi i(k'_x+k_x)}-1)}{k'_x+k_x} = \frac{\cos 2\pi(k'_x+k_x)+i\sin 2\pi(k'_x+k_x)-1}{k'_x+k_x} = 0, \tag{77}$$

where we used the fact that, as k_x and k'_x are integer numbers, k'_x+k_x are too. And so, if we combine the two answers, we have a kronecker delta and in eq. (74) we have

$$\int d^3x\left(a_{\boldsymbol{p'}}a_{\boldsymbol{p}}e^{-i[t(E'+E)-\boldsymbol{r}(\boldsymbol{p'}+\boldsymbol{p})]}\right) = a_{\boldsymbol{p'}}a_{\boldsymbol{p}}e^{-it(E'+E)}L^3\delta_{\boldsymbol{p'}+\boldsymbol{p},0}. \tag{78}$$

Now, if we do the same integration for the other terms of eq. (74), we have similar answers and we can generalize

$$\int d^3x e^{\pm i\boldsymbol{r}(\boldsymbol{p'}+\boldsymbol{p})]} = L^3\delta_{\boldsymbol{p'}+\boldsymbol{p},0}, \tag{79}$$

$$\int d^3x e^{\pm i\boldsymbol{r}(\boldsymbol{p'}-\boldsymbol{p})]} = L^3\delta_{\boldsymbol{p'}-\boldsymbol{p},0}. \tag{80}$$

And now, putting everything together, we have

$$\begin{aligned}\boldsymbol{P} &= -\int d^3x T_{0i} \\ &= \sum_{\boldsymbol{p'}}\sum_{\boldsymbol{p}} \frac{E'\boldsymbol{p}}{\sqrt{4E'EV^2}}\Big(- a_{\boldsymbol{p'}}a_{\boldsymbol{p}}e^{-it(E'+E)}L^3\delta_{\boldsymbol{p'}+\boldsymbol{p},0} - a^*_{\boldsymbol{p'}}a^*_{\boldsymbol{p}}e^{it(E'+E)}L^3\delta_{\boldsymbol{p'}+\boldsymbol{p},0} \\ &+ a_{\boldsymbol{p'}}a^*_{\boldsymbol{p}}e^{-it(E'-E)}L^3\delta_{\boldsymbol{p'}-\boldsymbol{p},0} + a^*_{\boldsymbol{p'}}a_{\boldsymbol{p}}e^{it(E'-E)})L^3\delta_{\boldsymbol{p'}-\boldsymbol{p},0}\Big) \\ &= \sum_{\boldsymbol{p}}\frac{\boldsymbol{p}}{2}\Big(a_{\boldsymbol{p}}a_{-\boldsymbol{p}}e^{-2iEt} + a^*_{\boldsymbol{p}}a^*_{-\boldsymbol{p}}e^{2iEt} + a_{\boldsymbol{p}}a^*_{\boldsymbol{p}} + a^*_{\boldsymbol{p}}a_{\boldsymbol{p}}\Big).\end{aligned} \tag{81}$$

Analyzing separately the first term of this summation:

$$\begin{aligned}\sum_{\boldsymbol{p}}\frac{\boldsymbol{p}}{2}(a_{\boldsymbol{p}}a_{-\boldsymbol{p}})e^{-2iEt} &= \sum_{\boldsymbol{p}<0}\frac{\boldsymbol{p}}{2}(a_{\boldsymbol{p}}a_{-\boldsymbol{p}})e^{-2iEt} + \sum_{\boldsymbol{p}>0}\frac{\boldsymbol{p}}{2}(a_{\boldsymbol{p}}a_{-\boldsymbol{p}})e^{-2iEt} \\ &= \sum_{\boldsymbol{p}>0}\frac{-\boldsymbol{p}}{2}(a_{-\boldsymbol{p}}a_{\boldsymbol{p}})e^{-2iEt} + \sum_{\boldsymbol{p}>0}\frac{\boldsymbol{p}}{2}(a_{\boldsymbol{p}}a_{-\boldsymbol{p}}e^{-2iEt}) \\ &= \sum_{\boldsymbol{p}>0}\frac{\boldsymbol{p}}{2}(a_{\boldsymbol{p}}a_{-\boldsymbol{p}} - a_{-\boldsymbol{p}}a_{\boldsymbol{p}})e^{-2iEt} = 0.\end{aligned} \tag{82}$$

For the second term the calculations are similar. And so we finally have the final form of the momentum

$$\boldsymbol{P} = \sum_{\boldsymbol{p}}\frac{\boldsymbol{p}}{2}(a_{\boldsymbol{p}}a^*_{\boldsymbol{p}} + a^*_{\boldsymbol{p}}a_{\boldsymbol{p}}). \tag{83}$$

This result is in accordance with Eq.(4.51) from [5].

H. Calculation of the Energy

We have that the energy is given by

$$\mathcal{E} = P_0 = \int d^3x T_{00}. \tag{84}$$

We have already calculated T_{00} on eq. (52). So we have

$$\mathcal{E} = \int d^3x \frac{1}{2}\left[\dot{\phi}^2 + (\boldsymbol{\nabla}\phi)^2 + m^2\phi^2\right]. \tag{85}$$

Using ϕ as given by eq. (68), we first calculate each term separately

$$\begin{aligned}\int d^3x\dot{\phi}^2 &= \int d^3x \sum_{\boldsymbol{p'}}\sum_{\boldsymbol{p}}\frac{-E'E}{\sqrt{4E'EV^2}}\Big[a_{\boldsymbol{p'}}a_{\boldsymbol{p}}e^{-i[t(E'+E)-\boldsymbol{r}(\boldsymbol{p'}+\boldsymbol{p})]} + a^*_{\boldsymbol{p'}}a^*_{\boldsymbol{p}}e^{i[t(E'+E)-\boldsymbol{r}(\boldsymbol{p'}+\boldsymbol{p})]} \\ &- a_{\boldsymbol{p'}}a^*_{\boldsymbol{p}}e^{-i[t(E'-E)-\boldsymbol{r}(\boldsymbol{p'}-\boldsymbol{p})]} - a^*_{\boldsymbol{p'}}a_{\boldsymbol{p}}e^{i[t(E'-E)-\boldsymbol{r}(\boldsymbol{p'}-\boldsymbol{p})]}\Big] \\ &= \sum_{\boldsymbol{p}}\frac{-E}{2}\Big(a_{-\boldsymbol{p}}a_{\boldsymbol{p}}e^{-2iEt} + a^*_{-\boldsymbol{p}}a^*_{\boldsymbol{p}}e^{2iEt} - a_{\boldsymbol{p}}a^*_{\boldsymbol{p}} - a^*_{\boldsymbol{p}}a_{\boldsymbol{p}}\Big),\end{aligned} \tag{86}$$

$$\int d^3x(\boldsymbol{\nabla}\phi)^2 = \int d^3x \sum_{\boldsymbol{p}'}\sum_{\boldsymbol{p}} \frac{-\boldsymbol{p}'\boldsymbol{p}}{\sqrt{4E'EV^2}}\Big[a_{\boldsymbol{p}'}a_{\boldsymbol{p}}e^{-i[t(E'+E)-\boldsymbol{r}(\boldsymbol{p}'+\boldsymbol{p})]} + a^*_{\boldsymbol{p}'}a^*_{\boldsymbol{p}}e^{i[t(E'+E)-\boldsymbol{r}(\boldsymbol{p}'+\boldsymbol{p})]}$$
$$- a_{\boldsymbol{p}'}a^*_{\boldsymbol{p}}e^{-i[t(E'-E)-\boldsymbol{r}(\boldsymbol{p}'-\boldsymbol{p})]} - a^*_{\boldsymbol{p}'}a_{\boldsymbol{p}}e^{i[t(E'-E)-\boldsymbol{r}(\boldsymbol{p}'-\boldsymbol{p})]}\Big]$$
$$= \sum_{\boldsymbol{p}} \frac{\boldsymbol{p}^2}{2E}\Big(a_{-\boldsymbol{p}}a_{\boldsymbol{p}}e^{-2iEt} + a^*_{-\boldsymbol{p}}a^*_{\boldsymbol{p}}e^{2iEt} + a_{\boldsymbol{p}}a^*_{\boldsymbol{p}} + a^*_{\boldsymbol{p}}a_{\boldsymbol{p}}\Big), \tag{87}$$

$$\int d^3x m^2\phi^2 = \int d^3x \sum_{\boldsymbol{p}'}\sum_{\boldsymbol{p}} \frac{m^2}{\sqrt{4E'EV^2}}\Big[a_{\boldsymbol{p}'}a_{\boldsymbol{p}}e^{-i[t(E'+E)-\boldsymbol{r}(\boldsymbol{p}'+\boldsymbol{p})]} + a^*_{\boldsymbol{p}'}a^*_{\boldsymbol{p}}e^{i[t(E'+E)-\boldsymbol{r}(\boldsymbol{p}'+\boldsymbol{p})]}$$
$$+ a_{\boldsymbol{p}'}a^*_{\boldsymbol{p}}e^{-i[t(E'-E)-\boldsymbol{r}(\boldsymbol{p}'-\boldsymbol{p})]} + a^*_{\boldsymbol{p}'}a_{\boldsymbol{p}}e^{i[t(E'-E)-\boldsymbol{r}(\boldsymbol{p}'-\boldsymbol{p})]}\Big]$$
$$= \sum_{\boldsymbol{p}} \frac{m^2}{2E}\Big(a_{-\boldsymbol{p}}a_{\boldsymbol{p}}e^{-2iEt} + a^*_{-\boldsymbol{p}}a^*_{\boldsymbol{p}}e^{2iEt} + a_{\boldsymbol{p}}a^*_{\boldsymbol{p}} + a^*_{\boldsymbol{p}}a_{\boldsymbol{p}}\Big). \tag{88}$$

Substituting this into eq. (85), we get

$$\mathcal{E} = \frac{1}{2}\sum_{\boldsymbol{p}}\Big[a_{-\boldsymbol{p}}a_{\boldsymbol{p}}e^{-2iEt}\left(\frac{-E}{2} + \frac{\boldsymbol{p}^2}{2E} + \frac{m^2}{2E}\right) + a^*_{-\boldsymbol{p}}a^*_{\boldsymbol{p}}e^{2iEt}\left(\frac{-E}{2} + \frac{\boldsymbol{p}^2}{2E} + \frac{m^2}{2E}\right)$$
$$+ a_{\boldsymbol{p}}a^*_{\boldsymbol{p}}\left(\frac{E}{2} + \frac{\boldsymbol{p}^2}{2E} + \frac{m^2}{2E}\right) + a^*_{\boldsymbol{p}}a_{\boldsymbol{p}}\left(\frac{E}{2} + \frac{\boldsymbol{p}^2}{2E} + \frac{m^2}{2E}\right)\Big]. \tag{89}$$

Taking a look at the terms in the brackets

$$\left(\frac{-E}{2} + \frac{\boldsymbol{p}^2}{2E} + \frac{m^2}{2E}\right) = \frac{-E^2 + \boldsymbol{p}^2 + m^2}{2E} = 0 \tag{90}$$

and

$$\left(\frac{E}{2} + \frac{\boldsymbol{p}^2}{2E} + \frac{m^2}{2E}\right) = \frac{E^2 + \boldsymbol{p}^2 + m^2}{2E} = E, \tag{91}$$

where we used the relativistic dispersion relation as in eq. (65). And so we finally have the final form of the energy

$$\mathcal{E} = \sum_{\boldsymbol{p}} \frac{E}{2}(a_{\boldsymbol{p}}a^*_{\boldsymbol{p}} + a^*_{\boldsymbol{p}}a_{\boldsymbol{p}}). \tag{92}$$

I. Calculation of the Noether charge

In order to calculate the Noether charge we need the current density given by eq.(61) and the relation [5]

$$Q = \int d^3x j^0(x). \tag{93}$$

The zero component of the current density is

$$j^0 = ie(\phi\partial^0\phi^* - \phi^*\partial^0\phi). \tag{94}$$

Using eq. (67) and doing the derivations, we have

$$\int d^3x\phi\partial_0\phi^* = \int d^3x \sum_{\boldsymbol{p}}\sum_{\boldsymbol{p'}} \frac{iE'}{\sqrt{4E'EV^2}}\Big[a_{\boldsymbol{p}}a^*_{\boldsymbol{p'}}e^{-i[t(E-E')-\boldsymbol{r}(\boldsymbol{p}-\boldsymbol{p'})]} - b_{\boldsymbol{p}}b^*_{\boldsymbol{p'}}e^{i[t(E-E')-\boldsymbol{r}(\boldsymbol{p}-\boldsymbol{p'})]}$$
$$- a_{\boldsymbol{p}}b^*_{\boldsymbol{p'}}e^{-i[t(E+E')-\boldsymbol{r}(\boldsymbol{p}-\boldsymbol{p'})]} + b_{\boldsymbol{p}}a^*_{\boldsymbol{p'}}e^{i[t(E+E')-\boldsymbol{r}(\boldsymbol{p}+\boldsymbol{p'})]}\Big]$$
$$= \sum_{\boldsymbol{p}} \frac{i}{2}\Big(a_{\boldsymbol{p}}a^*_{\boldsymbol{p}} - b_{\boldsymbol{p}}b^*_{\boldsymbol{p}} - a_{\boldsymbol{p}}b^*_{-\boldsymbol{p}}e^{-2iEt} + b_{\boldsymbol{p}}a^*_{-\boldsymbol{p}}e^{2iEt}\Big), \tag{95}$$

$$\int d^3x\phi^*\partial_0\phi = \int d^3x \sum_{\boldsymbol{p}}\sum_{\boldsymbol{p'}} \frac{-iE'}{\sqrt{4E'EV^2}}\Big[a^*_{\boldsymbol{p}}a_{\boldsymbol{p'}}e^{i[t(E-E')-\boldsymbol{r}(\boldsymbol{p}-\boldsymbol{p'})]} - b^*_{\boldsymbol{p}}b_{\boldsymbol{p'}}e^{-i[t(E-E')-\boldsymbol{r}(\boldsymbol{p}-\boldsymbol{p'})]}$$
$$- a^*_{\boldsymbol{p}}b_{\boldsymbol{p'}}e^{i[t(E+E')-\boldsymbol{r}(\boldsymbol{p}-\boldsymbol{p'})]} + b^*_{\boldsymbol{p}}a_{\boldsymbol{p'}}e^{-i[t(E+E')-\boldsymbol{r}(\boldsymbol{p}+\boldsymbol{p'})]}\Big]$$
$$= \sum_{\boldsymbol{p}} \frac{-i}{2}\Big(a^*_{\boldsymbol{p}}a_{\boldsymbol{p}} - b^*_{\boldsymbol{p}}b_{\boldsymbol{p}} - a^*_{\boldsymbol{p}}b_{-\boldsymbol{p}}e^{2iEt} + b^*_{\boldsymbol{p}}a_{-\boldsymbol{p}}e^{-2iEt}\Big). \tag{96}$$

Substituting this into eq. (93), we obtain

$$Q = ie\Big[\sum_{\boldsymbol{p}} \frac{i}{2}\Big(a_{\boldsymbol{p}}a^*_{\boldsymbol{p}} - b_{\boldsymbol{p}}b^*_{\boldsymbol{p}} - a_{\boldsymbol{p}}b^*_{-\boldsymbol{p}}e^{-2iEt} + b_{\boldsymbol{p}}a^*_{-\boldsymbol{p}}e^{2iEt}\Big)$$
$$- \sum_{\boldsymbol{p}} \frac{-i}{2}\Big(a^*_{\boldsymbol{p}}a_{\boldsymbol{p}} - b^*_{\boldsymbol{p}}b_{\boldsymbol{p}} - a^*_{\boldsymbol{p}}b_{-\boldsymbol{p}}e^{2iEt} + b^*_{\boldsymbol{p}}a_{-\boldsymbol{p}}e^{-2iEt}\Big)\Big]. \tag{97}$$

Looking at the therms multiplied by e^{2iEt}, we have

$$\sum_{\boldsymbol{p}} b_{\boldsymbol{p}}a^*_{-\boldsymbol{p}}e^{2iEt} - \sum_{\boldsymbol{p}} a^*_{\boldsymbol{p}}b_{-\boldsymbol{p}}e^{2iEt} = \sum_{\boldsymbol{p}<0}(b_{\boldsymbol{p}}a^*_{-\boldsymbol{p}} - a^*_{\boldsymbol{p}}b_{-\boldsymbol{p}})e^{2iEt}$$
$$+ \sum_{\boldsymbol{p}>0}(b_{\boldsymbol{p}}a^*_{-\boldsymbol{p}} - a^*_{\boldsymbol{p}}b_{-\boldsymbol{p}})e^{2iEt} + b_0a^*_0e^{2iEt} - a^*_0b_0e^{2iEt}$$
$$= \sum_{\boldsymbol{p}>0}\Big[(b_{-\boldsymbol{p}}a^*_{\boldsymbol{p}} - a^*_{-\boldsymbol{p}}b_{\boldsymbol{p}})e^{2iEt} + (b_{\boldsymbol{p}}a^*_{-\boldsymbol{p}} - a^*_{\boldsymbol{p}}b_{-\boldsymbol{p}})e^{2iEt}\Big] = 0. \tag{98}$$

The same happens with the terms multiplied by e^{-2iEt}. And so we finally have the final form of the Noether charge as

$$Q = -e\sum_{\boldsymbol{p}}(a_{\boldsymbol{p}}a^*_{\boldsymbol{p}} - b_{\boldsymbol{p}}b^*_{\boldsymbol{p}}). \tag{99}$$

J. Calculation of the Noether current

In order to calculate the Noether current we need the spatial part of the eq. (61)

$$j^i = ie(\phi\partial^i\phi^* - \phi^*\partial^i\phi) = ie[\phi^*(\boldsymbol{\nabla}\phi) - \phi(\boldsymbol{\nabla}\phi^*)]. \tag{100}$$

And we have

$$\boldsymbol{J} = ie\int d^3x[\phi^*(\boldsymbol{\nabla}\phi) - \phi(\boldsymbol{\nabla}\phi^*)]. \tag{101}$$

Doing the derivations and the integration of the two terms separately, we obtain

$$\int d^3x \phi^* (\boldsymbol{\nabla}\phi) = \int d^3x \sum_{\boldsymbol{p}} \sum_{\boldsymbol{p'}} \frac{i\boldsymbol{p'}}{\sqrt{4E'EV^2}} \Big[a^*_{\boldsymbol{p}} a_{\boldsymbol{p'}} e^{i[t(E-E')-\boldsymbol{r}(\boldsymbol{p}-\boldsymbol{p'})]} - b^*_{\boldsymbol{p}} b_{\boldsymbol{p'}} e^{-i[t(E-E')-\boldsymbol{r}(\boldsymbol{p}-\boldsymbol{p'})]}$$
$$- a^*_{\boldsymbol{p}} b_{\boldsymbol{p'}} e^{i[t(E+E')-\boldsymbol{r}(\boldsymbol{p}-\boldsymbol{p'})]} + b^*_{\boldsymbol{p}} a^*_{\boldsymbol{p'}} e^{-i[t(E+E')-\boldsymbol{r}(\boldsymbol{p}+\boldsymbol{p'})]} \Big]$$
$$= \sum_{\boldsymbol{p}} \frac{i\boldsymbol{p}}{2E} \Big(a^*_{\boldsymbol{p}} a_{\boldsymbol{p}} - b^*_{\boldsymbol{p}} b_{\boldsymbol{p}} + a^*_{\boldsymbol{p}} b_{-\boldsymbol{p}} e^{2iEt} - b^*_{\boldsymbol{p}} a_{-\boldsymbol{p}} e^{-2iEt} \Big), \tag{102}$$

$$\int d^3x \phi (\boldsymbol{\nabla}\phi^*) = \int d^3x \sum_{\boldsymbol{p}} \sum_{\boldsymbol{p'}} \frac{-i\boldsymbol{p'}}{\sqrt{4E'EV^2}} \Big[a_{\boldsymbol{p}} a^*_{\boldsymbol{p'}} e^{-i[t(E-E')-\boldsymbol{r}(\boldsymbol{p}-\boldsymbol{p'})]} - b_{\boldsymbol{p}} b^*_{\boldsymbol{p'}} e^{i[t(E-E')-\boldsymbol{r}(\boldsymbol{p}-\boldsymbol{p'})]}$$
$$- a_{\boldsymbol{p}} b^*_{\boldsymbol{p'}} e^{-i[t(E+E')-\boldsymbol{r}(\boldsymbol{p}-\boldsymbol{p'})]} + b_{\boldsymbol{p}} a^*_{\boldsymbol{p'}} e^{i[t(E+E')-\boldsymbol{r}(\boldsymbol{p}+\boldsymbol{p'})]} \Big]$$
$$= \sum_{\boldsymbol{p}} \frac{-i\boldsymbol{p}}{2E} \Big(a_{\boldsymbol{p}} a^*_{\boldsymbol{p}} - b_{\boldsymbol{p}} b^*_{\boldsymbol{p}} + a_{\boldsymbol{p}} b^*_{-\boldsymbol{p}} e^{-2iEt} - b_{\boldsymbol{p}} a^*_{-\boldsymbol{p}} e^{2iEt} \Big). \tag{103}$$

Noticing that the terms multiplying e^{2iEt} and e^{-2iEt} cancel each other as in the previous section, we finally have

$$\boldsymbol{J} = -e \sum_{\boldsymbol{p}} \frac{\boldsymbol{p}}{E} (a^*_{\boldsymbol{p}} a_{\boldsymbol{p}} - b^*_{\boldsymbol{p}} b_{\boldsymbol{p}}). \tag{104}$$

III. FIELD QUANTIZATION

We must observe that in high energy physics the number of particles of a given type is not a constant. So the interpretation of the Klein-Gordon equation as a single-particle equation has to be reformulated. If we take the field $\phi(x)$ as being a quantum field we will have a interpretation of a many-particle theory, which is what we want.
If we recognize $\phi(x)$ as a quantum field we should treated it as a operator, which is subject to various commutation relations analogous to those in ordinary quantum mechanics. So, let us start with the study of the ordinary quantum mechanic through the study of the Harmonic Oscillator.

A. One Harmonic Oscillator

The Hamiltonian operator of a one-dimensional harmonic oscillator is given by [12]

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2 \hat{x}^2}{2}. \tag{105}$$

Remembering that $\hat{p} = -i\hbar \frac{d}{dx}$. Now we suppose that

$$\hat{a} \equiv A\hat{x} + B\hat{p},$$
$$\hat{a}^\dagger \equiv A\hat{x} - B\hat{p}. \tag{106}$$

And we have

$$\hat{x} = \frac{\hat{a} + \hat{a}^\dagger}{2A},$$
$$\hat{p} = \frac{\hat{a} - \hat{a}^\dagger}{2B}. \tag{107}$$

Substituting this into eq. (105), we have

$$\hat{H} = (\hat{a}^\dagger)^2\left(\frac{m\omega^2}{8A^2} + \frac{1}{8mB^2}\right) + (\hat{a})^2\left(\frac{m\omega^2}{8A^2} + \frac{1}{8mB^2}\right) + (\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger)\left(\frac{m\omega^2}{8A^2} - \frac{1}{8mB^2}\right). \tag{108}$$

We want that the Hamiltonian of the harmonic oscillator has the following form

$$\hat{H} = \frac{\hbar\omega}{2}(\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger). \tag{109}$$

Comparing eq. (108) with eq. (109) we can see that the terms multiplying $(\hat{a}^\dagger)^2$ and $(\hat{a})^2$ have to be zero and the term multiplying $(\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger)$ have to be equal to $\frac{\hbar\omega}{2}$. With that we have that

$$\begin{aligned} A &= \sqrt{\frac{m\omega}{2\hbar}}, \\ B &= \frac{i}{\sqrt{2m\hbar\omega}}. \end{aligned} \tag{110}$$

Replacing it into eq. (106), we have

$$\begin{aligned} \hat{a} &= \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i\hat{p}}{m\omega}\right), \\ \hat{a}^\dagger &= \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i\hat{p}}{m\omega}\right) \end{aligned} \tag{111}$$

and into eq. (107)

$$\begin{aligned} \hat{x} &= \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}^\dagger + \hat{a}), \\ \hat{p} &= i\sqrt{\frac{m\hbar\omega}{2}}(\hat{a}^\dagger - \hat{a}). \end{aligned} \tag{112}$$

Eqs. (111) and (112) are in accordance with Eqs. (2.3.2) and (2.3.24) from Ref.[7].

Now we define a new variable ξ in order to facilitate our calculations further on

$$\xi \equiv \frac{x}{x_0}, \tag{113}$$

where $x_0 = \sqrt{\frac{\hbar}{m\omega}}$. With that we also have

$$\frac{d}{d\xi} = x_0\frac{d}{dx} \rightarrow \frac{d^2}{d\xi^2} = x_0^2\frac{d^2}{dx^2}. \tag{114}$$

Making this substitution into eq. (105), we now have the following Hamiltonian

$$\hat{H} = \frac{\hbar\omega}{2}\left(\xi^2 - \frac{d^2}{d\xi^2}\right). \tag{115}$$

Making also this substitution into eq. (111), we have

$$\begin{aligned} \hat{a} &= \frac{1}{\sqrt{2}}\left(\xi + \frac{d}{d\xi}\right), \\ \hat{a}^\dagger &= \frac{1}{\sqrt{2}}\left(\xi - \frac{d}{d\xi}\right). \end{aligned} \tag{116}$$

With this definition of $\hat{a}$ and $\hat{a}^\dagger$ we are also able to find

$$\begin{aligned}\xi &= \frac{1}{\sqrt{2}}(\hat{a}^\dagger + \hat{a}),\\ \frac{d}{d\xi} &= \frac{1}{\sqrt{2}}(\hat{a}^\dagger - \hat{a}).\end{aligned} \tag{117}$$

Using the definition of $\hat{a}$ and $\hat{a}^\dagger$ as given in eq. (116) we are now ready to calculate the commutation relations of this two operators

$$\begin{aligned}[\hat{a}, \hat{a}^\dagger] &= \left[\frac{1}{\sqrt{2}}\left(\xi + \frac{d}{d\xi}\right), \frac{1}{\sqrt{2}}\left(\xi - \frac{d}{d\xi}\right)\right]\\ &= \frac{1}{2}\left([\xi, \xi] - \left[\frac{d}{d\xi}, \frac{d}{d\xi}\right] + \left[\frac{d}{d\xi}, \xi\right] - \left[\xi, \frac{d}{d\xi}\right]\right).\end{aligned} \tag{118}$$

It is not hard to see that the first two commutators are equal to zero. To calculate the last two commutators it is easier to apply them to a dummy function ψ:

$$\begin{aligned}\left[\frac{d}{d\xi}, \xi\right]\psi - \left[\xi, \frac{d}{d\xi}\right]\psi &= \frac{d}{d\xi}(\xi\psi) - \xi\frac{d}{d\xi}\psi - \left(\xi\frac{d}{d\xi}\psi - \frac{d}{d\xi}(\xi\psi)\right)\\ &= 2\left(\frac{d}{d\xi}(\xi\psi) - \xi\frac{d}{d\xi}\psi\right)\\ &= 2\left(\psi + \xi\frac{d\psi}{d\xi} - \xi\frac{d}{d\xi}\psi\right)\\ &= 2\psi.\end{aligned} \tag{119}$$

Substituting this result, without the dummy function ψ, into eq. (118), we now have the final result [8]

$$[\hat{a}, \hat{a}^\dagger] = -[\hat{a}^\dagger, \hat{a}] = 1. \tag{120}$$

Doing similar calculations, we also have

$$[\hat{a}, \hat{a}] = [\hat{a}^\dagger, \hat{a}^\dagger] = 0. \tag{121}$$

We can also calculate the commutator between the Hamiltonian given by eq. (115) and $a^\dagger$:

$$\begin{aligned}[\hat{H}, \hat{a}^\dagger] &= \frac{\hbar\omega}{2\sqrt{2}}\left[\xi^2 - \frac{d^2}{d\xi^2}, \xi - \frac{d}{d\xi}\right]\\ &= \frac{\hbar\omega}{2\sqrt{2}}\left([\xi^2, \xi] + \left[\frac{d^2}{d\xi^2}, \frac{d}{d\xi}\right] - \left[\frac{d^2}{d\xi^2}, \xi\right] - \left[\xi^2, \frac{d}{d\xi}\right]\right).\end{aligned} \tag{122}$$

Again, the first two commutators are equal to zero. Let's calculate the other two

$$\begin{aligned}-\left[\frac{d^2}{d\xi^2}, \xi\right]\psi - \left[\xi^2, \frac{d}{d\xi}\right]\psi &= -\frac{d^2}{d\xi^2}(\xi\psi) + \xi\frac{d^2}{d\xi^2}\psi - \xi^2\frac{d}{d\xi}\psi + \frac{d}{d\xi}(\xi^2\psi)\\ &= -\frac{d}{d\xi}\left(\psi + \xi\frac{d}{d\xi}\psi\right) + \xi\frac{d^2}{d\xi^2}\psi - \xi^2\frac{d}{d\xi}\psi + 2\xi\psi + \xi^2\frac{d}{d\xi}\psi\\ &= 2\xi\psi - 2\frac{d}{d\xi}\psi\\ &= 2\sqrt{2}\hat{a}^\dagger\psi.\end{aligned} \tag{123}$$

And so we finally have [8]

$$[\hat{H}, \hat{a}^\dagger] = \hbar\omega\hat{a}^\dagger. \tag{124}$$

If we do similar calculations using $\hat{a}$ instead of $\hat{a}^\dagger$ we find the following

$$[\hat{H}, \hat{a}] = -\hbar\omega\hat{a}. \tag{125}$$

The $\hat{a}$ and $\hat{a}^\dagger$ operators are called annihilation (or lowering) and creation (or raising) operators, respectively. Let's see why. Assuming that ψ_n is an eigenstate of the Hamiltonian $\hat{H}\psi_n = E_n\psi_n$ and using the above commutation relations, it follows that

$$\hat{H}\hat{a}\psi_n = (\hat{a}\hat{H} - \hbar\omega\hat{a})\psi_n = (E_n - \hbar\omega)\hat{a}\psi_n, \tag{126}$$

$$\hat{H}\hat{a}^\dagger\psi_n = (\hat{a}^\dagger\hat{H} + \hbar\omega\hat{a}^\dagger)\psi_n = (E_n + \hbar\omega)\hat{a}^\dagger\psi_n. \tag{127}$$

This shows that $\hat{a}\psi_n$ and $\hat{a}^\dagger\psi_n$ are also eigenstates of the Hamiltonian, with eigenvalues $E_n - \hbar\omega$ and $E_n + \hbar\omega$ respectively. This identifies the operators $\hat{a}$ and $\hat{a}^\dagger$ as "lowering" and "raising" operators between adjacent eigenstates.

The ground state can be found by assuming that $\hat{a}\psi_0 = 0$, with $\psi_0 \neq 0$. Using $\hat{a}$ given by eq. (116) we have

$$\frac{1}{\sqrt{2}}\left(\xi + \frac{d}{d\xi}\right)\psi_0 = 0 \rightarrow \xi\psi_0 + \frac{d\psi_0}{d\xi} = 0 \tag{128}$$

which gives us

$$\psi_0(\xi) = Ae^{-\frac{\xi^2}{2}} \qquad \rightarrow \qquad \psi_0(x) = Ae^{-\frac{x^2}{2x_0^2}}, \tag{129}$$

where A is a normalizing constant that can be found by doing

$$\int_{-\infty}^{\infty} dx|\psi_0|^2 = |A|^2\int_{-\infty}^{\infty} dxe^{-\frac{x^2}{x_0^2}} = |A|^2\sqrt{\pi x_0^2} = 1 \qquad \rightarrow \qquad A = \frac{1}{(\pi x_0^2)^{1/4}} \tag{130}$$

and we finally have

$$\psi_0(x) = \frac{1}{(\pi x_0^2)^{1/4}}e^{-\frac{x^2}{2x_0^2}} = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}. \tag{131}$$

This result for the ground state wave function is in accordance with Eq.(2.3.30) from Ref.[7].

Now that we know that $\hat{a}^\dagger$ is a creation operator, it's intuitive to think that $(\hat{a}^\dagger)^n\psi_0 = C_n\psi_n$, where C_n is a proportionality factor. Let us find this factor!

First, let us introduce the Bra-Ket notation here. In this notation we have $\psi_n = |n\rangle$ and $\psi_n^* = \langle n|$. And let us redefine C_n as $C_n \rightarrow (C_n)^{-1}$ so that we have

$$|n\rangle = C_n(\hat{a}^\dagger)^n |0\rangle \qquad and \qquad \langle n| = \langle 0| (\hat{a})^n C_n^*. \tag{132}$$

In order to find C_n we will use the normalization rule $\langle n|n\rangle = 1$, the commutation relation given by eq. (120) so that $\hat{a}\hat{a}^\dagger = 1 + \hat{a}^\dagger\hat{a}$ and the fact that $\hat{a}|0\rangle = 0$

$$\begin{aligned}\langle n|n\rangle &= \langle 0|\,(\hat{a})^n C_n^* C_n (\hat{a}^\dagger)^n\,|0\rangle = 1\\ &= |C_n|^2 \langle 0|\,(\hat{a})^n (\hat{a}^\dagger)^n\,|0\rangle = |C_n|^2 \langle 0|\,(\hat{a})^{n-1}\hat{a}\hat{a}^\dagger(\hat{a}^\dagger)^{n-1}\,|0\rangle\\ &= |C_n|^2 \langle 0|\,(\hat{a})^{n-1}(1+\hat{a}^\dagger\hat{a})(\hat{a}^\dagger)^{n-1}\,|0\rangle\\ &= |C_n|^2 \left[\langle 0|\,(\hat{a})^{n-1}(\hat{a}^\dagger)^{n-1}\,|0\rangle + \langle 0|\,(\hat{a})^{n-1}\hat{a}^\dagger\hat{a}(\hat{a}^\dagger)^{n-1}\,|0\rangle\right].\end{aligned} \tag{133}$$

At this step, if $n = 1$, we have

$$\langle 1|1\rangle = |C_1|^2\left[\langle 0|0\rangle + \langle 0|\,\hat{a}^\dagger\hat{a}\,|0\rangle\right] = |C_1|^2. \tag{134}$$

Continuing the calculations of eq. (133)

$$\begin{aligned}\langle n|n\rangle &= |C_n|^2\left[\langle 0|\,(\hat{a})^{n-2}\hat{a}\hat{a}^\dagger(\hat{a}^\dagger)^{n-2}\,|0\rangle + \langle 0|\,(\hat{a})^{n-2}\hat{a}\hat{a}^\dagger\hat{a}\hat{a}^\dagger(\hat{a}^\dagger)^{n-2}\,|0\rangle\right]\\ &= |C_n|^2\left[\langle 0|\,(\hat{a})^{n-2}(1+\hat{a}^\dagger\hat{a})(\hat{a}^\dagger)^{n-2}\,|0\rangle + \langle 0|\,(\hat{a})^{n-2}(1+\hat{a}^\dagger\hat{a})(1+\hat{a}\hat{a}^\dagger)(\hat{a}^\dagger)^{n-2}\,|0\rangle\right]\\ &= |C_n|^2\left[2\,\langle 0|\,(\hat{a})^{n-2}(\hat{a}^\dagger)^{n-2}\,|0\rangle + 4\,\langle 0|\,(\hat{a})^{n-2}\hat{a}^\dagger\hat{a}(\hat{a}^\dagger)^{n-2}\,|0\rangle + \langle 0|\,(\hat{a})^{n-2}(\hat{a}^\dagger)^2(\hat{a})^2(\hat{a}^\dagger)^{n-2}\,|0\rangle\right].\end{aligned} \tag{135}$$

At this step, if we make $n = 2$, we have

$$\langle 2|2\rangle = |C_2|^2\left[2\,\langle 0|0\rangle + 4\,\langle 0|\,\hat{a}^\dagger\hat{a}\,|0\rangle + \langle 0|\,\hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a}\,|0\rangle\right] = 2|C_2|^2. \tag{136}$$

Continuing the calculations of eq. (135)

$$\begin{aligned}\langle n|n\rangle &= |C_n|^2\left[2\,\langle 0|\,(\hat{a})^{n-3}\hat{a}\hat{a}^\dagger(\hat{a}^\dagger)^{n-3}\,|0\rangle + 4\,\langle 0|\,(\hat{a})^{n-3}\hat{a}\hat{a}^\dagger\hat{a}\hat{a}^\dagger(\hat{a}^\dagger)^{n-3}\,|0\rangle + \langle 0|\,(\hat{a})^{n-3}\hat{a}(\hat{a}^\dagger)^2(\hat{a})^2\hat{a}^\dagger(\hat{a}^\dagger)^{n-3}\,|0\rangle\right]\\ &= |C_n|^2\Big[2\,\langle 0|\,(\hat{a})^{n-3}(1+\hat{a}^\dagger\hat{a})(\hat{a}^\dagger)^{n-3}\,|0\rangle + 4\,\langle 0|\,(\hat{a})^{n-3}(1+\hat{a}^\dagger\hat{a})(1+\hat{a}^\dagger\hat{a})(\hat{a}^\dagger)^{n-3}\,|0\rangle\\ &+ \langle 0|\,(\hat{a})^{n-3}(1+\hat{a}^\dagger\hat{a})(\hat{a}^\dagger)(\hat{a})(1+\hat{a}^\dagger\hat{a})(\hat{a}^\dagger)^{n-3}\,|0\rangle\Big]\\ &= |C_n|^2\Big[6\,\langle 0|\,(\hat{a})^{n-3}(\hat{a}^\dagger)^{n-3}\,|0\rangle + 18\,\langle 0|\,(\hat{a})^{n-3}\hat{a}^\dagger\hat{a}(\hat{a}^\dagger)^{n-3}\,|0\rangle\\ &+ 9\,\langle 0|\,(\hat{a})^{n-3}(\hat{a}^\dagger)^2(\hat{a})^2(\hat{a}^\dagger)^{n-3}\,|0\rangle + \langle 0|\,(\hat{a})^{n-3}(\hat{a}^\dagger)^3(\hat{a})^3(\hat{a}^\dagger)^{n-3}\,|0\rangle\Big].\end{aligned} \tag{137}$$

At this step, if we make $n = 3$, we have

$$\langle 3|3\rangle = |C_3|^2\left[6\,\langle 0|0\rangle + 18\,\langle 0|\,\hat{a}^\dagger\hat{a}\,|0\rangle + 9\,\langle 0|\,\hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a}\,|0\rangle + \langle 0|\,\hat{a}^\dagger\hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a}\hat{a}\,|0\rangle\right] = 6|C_3|^2 \tag{138}$$

and now, taking eqs: (134), (136) and (138), we can observe a pattern

$$\langle n|n\rangle = n!|C_n|^2 \tag{139}$$

which finally gives us the factor $C_n = 1/\sqrt{n!}$ that we were looking for and we have a very useful equation

$$|n\rangle = \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}}\,|0\rangle \qquad \leftrightarrow \qquad \psi_n = \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}}\psi_0, \tag{140}$$

which is equal to Eq.(2.3.21) from [7]. Using this equation and also $\hat{a}^\dagger$ given by eq. (116), we can find the general wave function of the harmonic oscillator as follows

$$\hat{a}^\dagger\psi_0 = \frac{1}{\sqrt{2}}\left(\xi - \frac{d}{d\xi}\right)\psi_0 = \frac{2}{\sqrt{2}}\xi\psi_0 = \frac{H_1(\xi)}{\sqrt{2}}\psi_0 = \psi_1$$

$$(\hat{a}^\dagger)^2\psi_0 = \left(\frac{1}{\sqrt{2}}\left(\xi - \frac{d}{d\xi}\right)\right)^2 \psi_0 = \frac{(4\xi^2 - 2)}{\sqrt{4}}\psi_0 = \frac{H_2(\xi)}{\sqrt{2^2}}\psi_0 = \sqrt{2}\psi_2$$

$$(\hat{a}^\dagger)^3\psi_0 = \left(\frac{1}{\sqrt{2}}\left(\xi - \frac{d}{d\xi}\right)\right)^3 \psi_0 = \frac{(8\xi^3 - 18\xi)}{\sqrt{8}}\psi_0 = \frac{H_3(\xi)}{\sqrt{2^3}}\psi_0 = \sqrt{3!}\psi_3$$

$$(\hat{a}^\dagger)^4\psi_0 = \left(\frac{1}{\sqrt{2}}\left(\xi - \frac{d}{d\xi}\right)\right)^4 \psi_0 = \frac{(16\xi^4 - 48\xi^2 + 12)}{\sqrt{16}}\psi_0 = \frac{H_4(\xi)}{\sqrt{2^4}}\psi_0 = \sqrt{4!}\psi_4$$

$$\vdots$$

$$(\hat{a}^\dagger)^n\psi_0 = \left(\frac{1}{\sqrt{2}}\left(\xi - \frac{d}{d\xi}\right)\right)^n \psi_0 = \frac{H_n(\xi)}{\sqrt{2^n}}\psi_0 = \sqrt{n!}\psi_n, \tag{141}$$

where $H_n(\xi)$ are the Hermite polynomials [11]:

$$H_n(\xi) = (-1)^n e^{\xi^2}\frac{d^n}{d\xi^n}e^{-\xi^2}. \tag{142}$$

And so we have the general solution for the Schrödinger Equation $\hat{H}\psi_n = E_n\psi_n$ from eqs. (131) and (141)

$$\psi_n(\xi) = \frac{1}{(\pi x_0^2)^{1/4}}\frac{1}{\sqrt{2^n n!}}e^{-\frac{\xi^2}{2}}H_n(\xi)$$

$$\psi_n(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\frac{1}{\sqrt{2^n n!}}e^{-\frac{m\omega}{2\hbar}x^2}H_n\left(x\sqrt{\frac{m\omega}{\hbar}}\right). \tag{143}$$

Let us take a look again at eq. (140). If we change n for $n+1$, we obtain

$$|n+1\rangle = \frac{(\hat{a}^\dagger)^{n+1}}{\sqrt{(n+1)!}}|0\rangle = \frac{\hat{a}^\dagger(\hat{a}^\dagger)^n}{\sqrt{(n+1)!}}|0\rangle = \frac{\hat{a}^\dagger\sqrt{n!}}{\sqrt{(n+1)!}}|n\rangle = \frac{\hat{a}^\dagger}{\sqrt{n+1}}|n\rangle \tag{144}$$

and so we have a well known recurrence relation that can be found in any references about Quantum Mechanics, as, for example, in Ref. [7]:

$$\hat{a}^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle\,. \tag{145}$$

Let us find a similar relation for the annihilation operator $\hat{a}$. Also from eq. (140), we have

$$\begin{aligned}
\hat{a}|n\rangle &= \frac{\hat{a}(\hat{a}^\dagger)^n}{\sqrt{n!}}|0\rangle = \frac{\hat{a}\hat{a}^\dagger(\hat{a}^\dagger)^{n-1}}{\sqrt{n!}}|0\rangle = \frac{(1+\hat{a}^\dagger\hat{a})(\hat{a}^\dagger)^{n-1}}{\sqrt{n!}}|0\rangle \\
&= \frac{(\hat{a}^\dagger)^{n-1}}{\sqrt{n!}}|0\rangle + \frac{\hat{a}^\dagger\hat{a}(\hat{a}^\dagger)^{n-1}}{\sqrt{n!}}|0\rangle = \frac{\sqrt{(n-1)!}}{\sqrt{n!}}|n-1\rangle + \frac{\hat{a}^\dagger\hat{a}\hat{a}^\dagger(\hat{a}^\dagger)^{n-2}}{\sqrt{n!}}|0\rangle \\
&= \frac{1}{\sqrt{n}}|n-1\rangle + \frac{\hat{a}^\dagger(1+\hat{a}^\dagger\hat{a})(\hat{a}^\dagger)^{n-2}}{\sqrt{n!}}|0\rangle = \frac{2}{\sqrt{n}}|n-1\rangle + \frac{\hat{a}^\dagger\hat{a}^\dagger\hat{a}(\hat{a}^\dagger)^{n-2}}{\sqrt{n!}}|0\rangle \\
&= \frac{2}{\sqrt{n}}|n-1\rangle + \frac{\hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a}^\dagger(\hat{a}^\dagger)^{n-3}}{\sqrt{n!}}|0\rangle = \frac{3}{\sqrt{n}}|n-1\rangle + \frac{(\hat{a}^\dagger)^3\hat{a}(\hat{a}^\dagger)^{n-3}}{\sqrt{n!}}|0\rangle
\end{aligned}$$

$$\vdots$$

$$= \frac{n}{\sqrt{n}} |n-1\rangle = \sqrt{n} |n-1\rangle . \tag{146}$$

And so we finally have another well known recurrence relation

$$\hat{a} |n\rangle = \sqrt{n} |n-1\rangle . \tag{147}$$

Observing eqs. (145) and (147) we can see again why the $\hat{a}^\dagger$ and $\hat{a}$ are called creation and annihilation operators, respectively: in essence, $\hat{a}^\dagger$ adds a single quantum of energy to the oscillator, while $\hat{a}$ removes a quantum. The two operators together are also called ladder operators.

We can also define a operator $\hat{N}$ called Number Operator

$$\hat{N} = \hat{a}^\dagger \hat{a}. \tag{148}$$

Let's apply this operator to a eigenstate $|n\rangle$

$$\begin{aligned} \hat{N} |n\rangle &= \hat{a}^\dagger \hat{a} |n\rangle = \hat{a}^\dagger \sqrt{n} |n-1\rangle = \sqrt{n} \hat{a}^\dagger |n-1\rangle \\ &= \sqrt{n}\sqrt{n-1+1} |n-1+1\rangle = n |n\rangle . \end{aligned} \tag{149}$$

In quantum mechanics, for systems where the total number of particles may not be preserved, the number operator is the observable that counts the number of particles.

With that and the commutation relation given by eq. (120) we can redefine the Hamiltonian given by eq. (109) as

$$\hat{H} = \hbar\omega \left(\hat{N} + \frac{1}{2} \right) . \tag{150}$$

Using the time-independent Schrödinger equation $\hat{H} |n\rangle = E_n |n\rangle$ we can also find the energy spectrum of the harmonic oscillator

$$\hat{H} |n\rangle = \hbar\omega \left(\hat{N} + \frac{1}{2} \right) |n\rangle = \hbar\omega \left(n + \frac{1}{2} \right) |n\rangle \tag{151}$$

which gives us

$$E_n = \hbar\omega \left(n + \frac{1}{2} \right) . \tag{152}$$

We can see that the energy is quantized, meaning that only discrete energy values are possible. Also, these discrete energy levels are equally spaced. Another thing that we can see is the fact that the lowest achievable energy is not equal to zero, but $\hbar\omega/2$.

We can also calculate the commutator of $\hat{N}$ with the creation and annihilation operators

$$[\hat{N}, \hat{a}^\dagger] = [\hat{a}^\dagger \hat{a}, \hat{a}^\dagger] = \hat{a}^\dagger [\hat{a}, \hat{a}^\dagger] + [\hat{a}^\dagger, \hat{a}^\dagger]\hat{a} = \hat{a}^\dagger, \tag{153}$$

$$[\hat{N}, \hat{a}] = [\hat{a}^\dagger \hat{a}, \hat{a}] = \hat{a}^\dagger [\hat{a}, \hat{a}] + [\hat{a}^\dagger, \hat{a}]\hat{a} = -\hat{a}, \tag{154}$$

where we used the commutation relations given by eqs. (120) and (121).

B. Many Harmonic Oscillators

We can generalize the results obtained above for one harmonic oscillator to a system of harmonic oscillators. For that, we take the momentum space as a three-dimensional lattice. So, every vertex contains one oscillator, but the spaces between the vertices are forbidden. The main results obtained at the section above are, as follows.

From eq. (120) and (121) we have

$$[\hat{a}_{\boldsymbol{p}}, \hat{a}^{\dagger}_{\boldsymbol{p}'}] = \delta_{\boldsymbol{p},\boldsymbol{p}'}, \tag{155}$$

$$[\hat{a}_{\boldsymbol{p}}, \hat{a}_{\boldsymbol{p}'}] = [\hat{a}^{\dagger}_{\boldsymbol{p}}, \hat{a}^{\dagger}_{\boldsymbol{p}'}] = 0. \tag{156}$$

We have N vertices and consequently N oscillators, with $N \to \infty$. So, from eq. (155) we can see that, indeed, the spaces between the vertices of the infinity three-dimensional momentum lattice are forbidden.

The action of the operator $\hat{a}_{\boldsymbol{p}}$ to the ground state here should also be zero

$$\hat{a}\,|0\rangle = 0 \qquad \Longleftrightarrow \qquad \hat{a}_{\boldsymbol{p}}\,|00...0\rangle = 0. \tag{157}$$

From eq. (140)

$$|n\rangle = \frac{(\hat{a}^{\dagger})^n}{\sqrt{n!}}\,|0\rangle \qquad \Longleftrightarrow \qquad \left|n_{\boldsymbol{p}_1} n_{\boldsymbol{p}_2} ... n_{\boldsymbol{p}_N}\right\rangle = \prod_{i=1}^{N} \frac{(\hat{a}^{\dagger}_{\boldsymbol{p}_i})^{n_{\boldsymbol{p}_i}}}{\sqrt{n_{\boldsymbol{p}_i}!}}\,|00...0\rangle\,. \tag{158}$$

From eqs. (145) and (147)

$$\hat{a}^{\dagger}_{\boldsymbol{p}}\left|n_{\boldsymbol{p}_1} \ldots n_{\boldsymbol{p}} \ldots n_{\boldsymbol{p}_N}\right\rangle = \sqrt{n_{\boldsymbol{p}}+1}\left|n_{\boldsymbol{p}_1} \ldots n_{\boldsymbol{p}}+1 \ldots n_{\boldsymbol{p}_N}\right\rangle \tag{159}$$

and

$$\hat{a}_{\boldsymbol{p}}\left|n_{\boldsymbol{p}_1} \ldots n_{\boldsymbol{p}} \ldots n_{\boldsymbol{p}_N}\right\rangle = \sqrt{n_{\boldsymbol{p}}}\left|n_{\boldsymbol{p}_1} \ldots n_{\boldsymbol{p}}-1 \ldots n_{\boldsymbol{p}_N}\right\rangle. \tag{160}$$

From eq. (148) we now have

$$\hat{N}_{\boldsymbol{p}} = \hat{a}^{\dagger}_{\boldsymbol{p}}\hat{a}_{\boldsymbol{p}} \tag{161}$$

so that

$$\hat{N}_{\boldsymbol{p}}\left|n_{\boldsymbol{p}_1} \ldots n_{\boldsymbol{p}} \ldots n_{\boldsymbol{p}_N}\right\rangle = n_{\boldsymbol{p}}\left|n_{\boldsymbol{p}_1} \ldots n_{\boldsymbol{p}} \ldots n_{\boldsymbol{p}_N}\right\rangle. \tag{162}$$

Here we have that $n_{\boldsymbol{p}}$ represents the number of particles at the position $\boldsymbol{p}$ in the lattice of the momentum space. So, the number operator $\hat{N}_{\boldsymbol{p}}$ counts this particles. And $\hat{a}^{\dagger}$ creates and $\hat{a}$ destroys particles.

C. Canonical Quantization of the Klein-Gordon Field

Now that we exceeded the study of the harmonic oscillator, we can finally use this knowledge for the quantization of the Klein-Gordon Field. We follow the prescription of field quantization: the field $\phi(x)$, given by eq. (67) is replaced by the operator $\hat{\phi}(x)$ as well as the amplitudes a and a^*, that are replaced by the annihilation and creation operators, so that we have

$$\hat{\phi}(x) = \sum_{\boldsymbol{p}} \frac{1}{\sqrt{2EV}}\left(\hat{a}_{\boldsymbol{p}}e^{-ipx} + \hat{a}^{\dagger}_{\boldsymbol{p}}e^{ipx}\right). \tag{163}$$

And all the other results obtained at section (II) may now be rewritten in terms of operators.

From the equation of the momentum, given by eq. (84), we have

$$\begin{aligned}\hat{\boldsymbol{P}} &= \sum_{\boldsymbol{p}} \frac{\boldsymbol{p}}{2}\Big(\hat{a}_{\boldsymbol{p}}\hat{a}^{\dagger}_{\boldsymbol{p}} + \hat{a}^{\dagger}_{\boldsymbol{p}}\hat{a}_{\boldsymbol{p}}\Big) \\ &= \sum_{\boldsymbol{p}} \boldsymbol{p}\Big(\hat{N}_{\boldsymbol{p}} + \frac{1}{2}\Big), \end{aligned} \tag{164}$$

where we used eqs. (155) and (161).

From eq.(92) we also have the Hamiltonian operator

$$\begin{aligned}\hat{H} &= \sum_{\boldsymbol{p}} \frac{E}{2}\Big(\hat{a}_{\boldsymbol{p}}\hat{a}^{\dagger}_{\boldsymbol{p}} + \hat{a}^{\dagger}_{\boldsymbol{p}}\hat{a}_{\boldsymbol{p}}\Big) \\ &= \sum_{\boldsymbol{p}} E\Big(\hat{N}_{\boldsymbol{p}} + \frac{1}{2}\Big), \end{aligned} \tag{165}$$

which is very similar to the Hamiltonian of the harmonic oscillator as seen in eq. (150) if we take into account that for the H.O. $E = \hbar\omega$. For the Klein-Gordon Field, as we already deduced, $E = \sqrt{\boldsymbol{p}^2 + m^2}$.

Taking the Klein-Gordon Field given by eq. (68) and using the following identity to replace the discrete sum by a continuous integral

$$\sum_{\boldsymbol{p}} \Longleftrightarrow \frac{V}{(2\pi)^3}\int d^3p \tag{166}$$

we have

$$\hat{\phi}(\boldsymbol{x}, t) = \frac{V}{(2\pi)^3}\int \frac{d^3p}{\sqrt{2EV}}\left(\hat{a}_{\boldsymbol{p}} e^{-i(Et-\boldsymbol{px})} + \hat{a}^{\dagger}_{\boldsymbol{p}} e^{i(Et-\boldsymbol{px})}\right). \tag{167}$$

Remembering that $\dot{\phi}(x) = \pi(x)$

$$\hat{\pi}(\boldsymbol{x}, t) = \frac{V}{(2\pi)^3}\int \frac{d^3p}{\sqrt{2EV}}(-iE)\left(\hat{a}_{\boldsymbol{p}} e^{-i(Et-\boldsymbol{px})} - \hat{a}^{\dagger}_{\boldsymbol{p}} e^{i(Et-\boldsymbol{px})}\right). \tag{168}$$

And we can calculate the commutation relations between $\hat{\phi}(\boldsymbol{x}, t)$ and $\hat{\pi}(\boldsymbol{x}, t)$

$$\begin{aligned}[\hat{\phi}(\boldsymbol{x}, t), \hat{\pi}(\boldsymbol{x'}, t)] = \frac{V}{2(2\pi)^6}\int d^3p \int d^3p' \frac{(-iE')}{\sqrt{4EE'}}\Bigg[&[\hat{a}_{\boldsymbol{p}}, \hat{a}_{\boldsymbol{p'}}]e^{-i(px+p'x')} - [\hat{a}^{\dagger}_{\boldsymbol{p}}, \hat{a}^{\dagger}_{\boldsymbol{p'}}]e^{i(px+p'x')} \\ &- [\hat{a}_{\boldsymbol{p}}, \hat{a}^{\dagger}_{\boldsymbol{p'}}]e^{-i(px-p'x')} + [\hat{a}^{\dagger}_{\boldsymbol{p}}, \hat{a}_{\boldsymbol{p'}}]e^{i(px-p'x')}\Bigg]. \end{aligned} \tag{169}$$

Using the commutation relations given by eqs.(155) and (156) we can see that the first two commutator are zero, resulting in

$$\begin{aligned}[\hat{\phi}(\boldsymbol{x}, t), \hat{\pi}(\boldsymbol{x'}, t)] &= \frac{V}{2(2\pi)^6}\int d^3p \int d^3p' \frac{(iE')}{\sqrt{4EE'}}\delta_{\boldsymbol{p},\boldsymbol{p'}}\left[e^{-i(px-p'x')} + e^{i(px-p'x')}\right] \\ &= \frac{iV}{2(2\pi)^6}\int d^3p \left[e^{i\boldsymbol{p}(\boldsymbol{x}-\boldsymbol{x'})} + e^{-i\boldsymbol{p}(\boldsymbol{x}-\boldsymbol{x'})}\right] = \frac{iV}{(2\pi)^3}\delta^3(\boldsymbol{x}-\boldsymbol{x'}), \end{aligned} \tag{170}$$

where we used the definition of Dirac delta function [9]

$$\delta^3(\boldsymbol{x}-\boldsymbol{x'}) = \int \frac{d^3p}{(2\pi)^3} e^{\pm i\boldsymbol{p}(\boldsymbol{x}-\boldsymbol{x'})}. \tag{171}$$

Doing similar calculations, we find

$$[\hat{\phi}(\boldsymbol{x},t), \hat{\phi}(\boldsymbol{x'},t)] = [\hat{\pi}(\boldsymbol{x},t), \hat{\pi}(\boldsymbol{x'},t)] = 0. \tag{172}$$

The commutation relations given by eqs. (170) and (172) are called equal-time commutation relations (ETCR) [5].

D. The Annihilation and Creation Operators in terms of the Klein-Gordon Field

In this section we want to find the annihilation an creation operators, $\hat{a}$ and $\hat{a}^\dagger$ as a function of the Klein-Gordon field $\phi(x)$. For that we will need the Fourier Transformations [10]:

$$F(\omega) = \int f(t)e^{-i\omega t}dt, \tag{173}$$

$$f(t) = \int \frac{1}{2\pi} F(\omega)e^{i\omega t}d\omega. \tag{174}$$

Now, if we separate eq. (167) into two equations, we have

$$\hat{\phi}(\boldsymbol{x},t) = \frac{V}{(2\pi)^3} \int \frac{d^3p}{\sqrt{2EV}} \left(\hat{a}_{\boldsymbol{p}} e^{-i(Et-\boldsymbol{px})}\right). \tag{175}$$

And, following the "recipe" given by eqs. (173) and (174), we do the Fourier transformation

$$\hat{a}_{\boldsymbol{p}} = \frac{1}{(2\pi)^3} \int d^3x \frac{(2\pi)^3}{V} (\sqrt{2EV}) e^{i(Et-\boldsymbol{px})} \hat{\phi}(\boldsymbol{x},t) = \int d^3x \frac{2E}{\sqrt{2EV}} e^{i(Et-\boldsymbol{px})} \hat{\phi}(\boldsymbol{x},t). \tag{176}$$

For the second part of eq. (167) we have

$$\hat{\phi}(\boldsymbol{x},t) = \frac{V}{(2\pi)^3} \int \frac{d^3p}{\sqrt{2EV}} \left(\hat{a}^\dagger_{\boldsymbol{p}} e^{i(Et-\boldsymbol{px})}\right). \tag{177}$$

Its Fourier transformation is

$$\hat{a}^\dagger_{\boldsymbol{p}} = \frac{1}{(2\pi)^3} \int d^3x \frac{(2\pi)^3}{V} (\sqrt{2EV}) e^{-i(Et-\boldsymbol{px})} \hat{\phi}(\boldsymbol{x},t) = \int d^3x \frac{2E}{\sqrt{2EV}} e^{-i(Et-\boldsymbol{px})} \hat{\phi}(\boldsymbol{x},t). \tag{178}$$

If we define a new function $u_{\boldsymbol{p}}$ and its complex conjugate $u^*_{\boldsymbol{p}}$ as

$$u_{\boldsymbol{p}} = \frac{1}{\sqrt{2EV}} e^{i(Et-\boldsymbol{px})} \qquad \text{and} \qquad u^*_{\boldsymbol{p}} = \frac{1}{\sqrt{2EV}} e^{-i(Et-\boldsymbol{px})} \tag{179}$$

we can rewrite eqs. (176) and (178) as

$$\hat{a}_{\boldsymbol{p}} = 2E \int d^3x u^*_{\boldsymbol{p}} \hat{\phi}(\boldsymbol{x},t), \tag{180}$$

$$\hat{a}^\dagger_{\boldsymbol{p}} = 2E \int d^3x u_{\boldsymbol{p}} \hat{\phi}(\boldsymbol{x},t). \tag{181}$$

This last two equations are equivalent to Eq.(4.28) from Ref.[5].

IV. PARTITION FUNCTION

In this section we want to find the Partition Function of the real scalar field. The thermodynamics of the real scalar field of volume V, in contact with the heat and particle reservoir of temperature T and chemical potential μ we have the following partition function [6]:

$$\mathcal{Z} = Tr(e^{-\beta(\hat{H}-\mu\hat{Q})}) \tag{182}$$

Where $\beta = 1/k_B T$, with k_B being the Boltzmann constant and T the temperature. But, in our case, the charge $\hat{Q}$ is zero. So we are left with

$$\mathcal{Z} = Tr(e^{-\beta\hat{H}}) \tag{183}$$

But how do we find the trace of a exponential of a matrix? Let us find out.

A. The trace of a exponential of a matrix

Let us calculate $Tr(e^{-\alpha\hat{A}})$ where $\hat{A}$ is a matrix given by

$$\hat{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

We can make the expansion of our exponential, so that we have

$$Tr(e^{-\alpha\hat{A}}) = Tr\Big(I - \alpha\hat{A} + \frac{\alpha^2\hat{A}^2}{2!} + ...\Big), \tag{184}$$

where I is the identity matrix. The trace of I is clearly 3 and the trace of $\alpha\hat{A}$ is $\alpha(a_{11} + a_{22} + a_{33})$, but the trace of $(\alpha^2\hat{A}^2)/2$ is a little more complicated to obtain. First we have to multiply $\hat{A}$ by $\hat{A}$ and than we can take the trace. And finally we have

$$Tr(e^{-\alpha\hat{A}}) = \sum_{i=1}^{3}\Big(1 - \alpha a_{ii} + \frac{\alpha^2}{2}\sum_{j=1}^{3} a_{ij}a_{ji} + ...\Big). \tag{185}$$

We can observe that this result is almost the Taylor series of $e^{-\alpha a_{ii}}$. So, if we take a particular case of the matrix $\hat{A}$, where the non-diagonal elements are zero, which means $a_{ij} = \delta_{ij}a_{ii}$, we obtain

$$\begin{aligned} Tr(e^{-\alpha\hat{A}}) &= \sum_{i=1}^{3}\Big(1 - \alpha a_{ii} + \frac{\alpha^2}{2}\sum_{j=1}^{3}\delta_{ij}a_{ii}\delta_{ji}a_{ii} + ...\Big) \\ &= \sum_{i=1}^{3}\Big(1 - \alpha a_{ii} + \frac{\alpha^2}{2}(a_{ii})^2 + ...\Big) \\ &= \sum_{i=1}^{3} e^{-\alpha a_{ii}}. \end{aligned} \tag{186}$$

But can we use this result to calculate the partition function? Only if the non-diagonal matrix elements of our Hamiltonian are zero. Let's verify that.

B. Matrix Elements of the Hamiltonian Operator

The matrix elements of our Hamiltonian operator are given by $\left\langle n'_{\boldsymbol{p'}_1} n'_{\boldsymbol{p'}_2} ... n'_{\boldsymbol{p'}_N} \right| \hat{H} \left| n_{\boldsymbol{p}_1} n_{\boldsymbol{p}_2} ... n_{\boldsymbol{p}_N} \right\rangle$. Where $\hat{H}$ is as given by eq. (165). Let us first calculate the eigenvalues

$$\begin{aligned} \hat{H} \left| n_{\boldsymbol{p}_1} n_{\boldsymbol{p}_2} ... n_{\boldsymbol{p}_N} \right\rangle &= \sum_{\boldsymbol{p}=-\infty}^{\infty} E_{\boldsymbol{p}} \Big(\hat{N}_{\boldsymbol{p}} + \frac{1}{2} \Big) \left| n_{\boldsymbol{p}_1} n_{\boldsymbol{p}_2} ... n_{\boldsymbol{p}_N} \right\rangle \\ &= \sum_{\boldsymbol{p}=-\infty}^{\infty} E_{\boldsymbol{p}} \Big(n_{\boldsymbol{p}} + \frac{1}{2} \Big) \left| n_{\boldsymbol{p}_1} n_{\boldsymbol{p}_2} ... n_{\boldsymbol{p}_N} \right\rangle . \end{aligned} \tag{187}$$

So, we have that the eigenvalues, let us call them $E_{n_{\boldsymbol{p}}}$, are

$$E_{n_{\boldsymbol{p}}} = \sum_{\boldsymbol{p}=-\infty}^{\infty} E_{\boldsymbol{p}} \Big(n_{\boldsymbol{p}} + \frac{1}{2} \Big) = \sum_{\boldsymbol{p}=-\infty}^{\infty} E_{\boldsymbol{p}} n_{\boldsymbol{p}} + \frac{1}{2} \sum_{\boldsymbol{p}=-\infty}^{\infty} E_{\boldsymbol{p}}. \tag{188}$$

Remembering that $E_{\boldsymbol{p}} = \sqrt{\boldsymbol{p}^2 + m^2}$ we can see that the second term is divergent, this term is often referred to as the ground state energy, while the first term represents a real physical quantity that is, for every term of the sum we have the total energy of that vertex of the three-dimensional momentum lattice.

Now we can multiply eq. (187) by the eigenstate Bra and, as the eigenvalues that we found are numbers, we have

$$\left\langle n'_{\boldsymbol{p'}_1} n'_{\boldsymbol{p'}_2} ... n'_{\boldsymbol{p'}_N} \right| \hat{H} \left| n_{\boldsymbol{p}_1} n_{\boldsymbol{p}_2} ... n_{\boldsymbol{p}_N} \right\rangle = \sum_{\boldsymbol{p}=-\infty}^{\infty} E_{\boldsymbol{p}} \Big(n_{\boldsymbol{p}} + \frac{1}{2} \Big) \left\langle n'_{\boldsymbol{p'}_1} n'_{\boldsymbol{p'}_2} ... n'_{\boldsymbol{p'}_N} \middle| n_{\boldsymbol{p}_1} n_{\boldsymbol{p}_2} ... n_{\boldsymbol{p}_N} \right\rangle . \tag{189}$$

Now let's take a look at this Bra-Ket. We know that $\langle n' | n \rangle = \delta(n' - n)$, where $\delta(n' - n)$ is a Kronecker delta. But, in our case, the eigenstates also depend on the momentum $\boldsymbol{p}$. So we have $\left\langle n'_{\boldsymbol{p'}} \middle| n_{\boldsymbol{p}} \right\rangle = \delta(n'_{\boldsymbol{p'}} - n_{\boldsymbol{p}}) \delta(\boldsymbol{p'} - \boldsymbol{p})$ for every $n_{\boldsymbol{p}}$

$$\begin{aligned} \left\langle n'_{\boldsymbol{p'}_1} n'_{\boldsymbol{p'}_2} ... n'_{\boldsymbol{p'}_N} \middle| n_{\boldsymbol{p}_1} n_{\boldsymbol{p}_2} ... n_{\boldsymbol{p}_N} \right\rangle &= \left\langle n'_{\boldsymbol{p'}_1} \middle| n_{\boldsymbol{p}_1} \right\rangle \left\langle n'_{\boldsymbol{p'}_2} \middle| n_{\boldsymbol{p}_2} \right\rangle ... \left\langle n'_{\boldsymbol{p'}_N} \middle| n_{\boldsymbol{p}_N} \right\rangle \\ &= \delta(n'_{\boldsymbol{p'}_1} - n_{\boldsymbol{p}_1}) \delta(\boldsymbol{p'}_1 - \boldsymbol{p}_1) \delta(n'_{\boldsymbol{p'}_2} - n_{\boldsymbol{p}_2}) \delta(\boldsymbol{p'}_2 - \boldsymbol{p}_2) ... \delta(n'_{\boldsymbol{p'}_N} - n_{\boldsymbol{p}_N}) \delta(\boldsymbol{p'}_N - \boldsymbol{p}_N) \\ &= \prod_{i=1}^{N} \delta(n'_{\boldsymbol{p'}_i} - n_{\boldsymbol{p}_i}) \delta(\boldsymbol{p'}_i - \boldsymbol{p}_i). \end{aligned} \tag{190}$$

We finally have our matrix elements of the Hamiltonian operator

$$\left\langle n'_{\boldsymbol{p'}_1} n'_{\boldsymbol{p'}_2} ... n'_{\boldsymbol{p'}_N} \right| \hat{H} \left| n_{\boldsymbol{p}_1} n_{\boldsymbol{p}_2} ... n_{\boldsymbol{p}_N} \right\rangle = \sum_{\boldsymbol{p}=-\infty}^{\infty} E_{\boldsymbol{p}} \Big(n_{\boldsymbol{p}} + \frac{1}{2} \Big) \prod_{i=1}^{N} \delta(n'_{\boldsymbol{p'}_i} - n_{\boldsymbol{p}_i}) \delta(\boldsymbol{p'}_i - \boldsymbol{p}_i). \tag{191}$$

With that we can see that only the diagonal elements of this matrix are non-zero, which is exactly what we need to use the result obtained at the previous subsection. The diagonal elements are

$$\left\langle n_{\boldsymbol{p}_1} n_{\boldsymbol{p}_2} ... n_{\boldsymbol{p}_N} \right| \hat{H} \left| n_{\boldsymbol{p}_1} n_{\boldsymbol{p}_2} ... n_{\boldsymbol{p}_N} \right\rangle = \sum_{\boldsymbol{p}=-\infty}^{\infty} E_{\boldsymbol{p}} \Big(n_{\boldsymbol{p}} + \frac{1}{2} \Big). \tag{192}$$

C. The Partition Function of the Real Scalar Field

Now we have everything we need to calculate the partition function of the real scalar field:

$$\mathcal{Z} = Tr(e^{-\beta \hat{H}}) = \sum_{n_{\boldsymbol{p}_1}=0}^{\infty} \sum_{n_{\boldsymbol{p}_2}=0}^{\infty} ... \sum_{n_{\boldsymbol{p}_N}=0}^{\infty} \left\langle n_{\boldsymbol{p}_1} n_{\boldsymbol{p}_2} ... n_{\boldsymbol{p}_N} \right| e^{-\beta \hat{H}} \left| n_{\boldsymbol{p}_1} n_{\boldsymbol{p}_2} ... n_{\boldsymbol{p}_N} \right\rangle . \tag{193}$$

Let us ignore for now the summations and focus only on the Bra-Ket. As was done on eq. (184), the exponential here can also be expanded

$$\begin{aligned}
\left\langle n_{\boldsymbol{p}_1} n_{\boldsymbol{p}_2} ... n_{\boldsymbol{p}_N} \right| e^{-\beta \hat{H}} \left| n_{\boldsymbol{p}_1} n_{\boldsymbol{p}_2} ... n_{\boldsymbol{p}_N} \right\rangle &= \left\langle n_{\boldsymbol{p}_1} n_{\boldsymbol{p}_2} ... n_{\boldsymbol{p}_N} \right| I - \beta \hat{H} + \frac{\beta^2 \hat{H}^2}{2} + ... \left| n_{\boldsymbol{p}_1} n_{\boldsymbol{p}_2} ... n_{\boldsymbol{p}_N} \right\rangle \\
&= \left\langle n_{\boldsymbol{p}_1} n_{\boldsymbol{p}_2} ... n_{\boldsymbol{p}_N} \right| I - \beta E_{n_{\boldsymbol{p}}} + \frac{\beta^2 E_{n_{\boldsymbol{p}}}^2}{2} + ... \left| n_{\boldsymbol{p}_1} n_{\boldsymbol{p}_2} ... n_{\boldsymbol{p}_N} \right\rangle \\
&= I - \beta E_{n_{\boldsymbol{p}}} + \frac{\beta^2 E_{n_{\boldsymbol{p}}}^2}{2} + ... = e^{-\beta E_{n_{\boldsymbol{p}}}} \\
&= exp\Big[-\beta \sum_{\boldsymbol{p}=-\infty}^{\infty} E_{\boldsymbol{p}} \Big(n_{\boldsymbol{p}} + \frac{1}{2} \Big) \Big].
\end{aligned} \tag{194}$$

Going back to eq. (193), we have

$$\begin{aligned}
\mathcal{Z} = Tr(e^{-\beta \hat{H}}) &= \sum_{n_{\boldsymbol{p}_1}=0}^{\infty} \sum_{n_{\boldsymbol{p}_2}=0}^{\infty} ... \sum_{n_{\boldsymbol{p}_N}=0}^{\infty} exp\Big[-\beta \sum_{\boldsymbol{p}=-\infty}^{\infty} E_{\boldsymbol{p}} \Big(n_{\boldsymbol{p}} + \frac{1}{2} \Big) \Big] \\
&= \Big[\sum_{n_{\boldsymbol{p}_1}=0}^{\infty} exp\Big[-\beta E_{\boldsymbol{p}_1} \Big(n_{\boldsymbol{p}_1} + \frac{1}{2} \Big) \Big] \Big] \Big[\sum_{n_{\boldsymbol{p}_2}=0}^{\infty} exp\Big[-\beta E_{\boldsymbol{p}_2} \Big(n_{\boldsymbol{p}_2} + \frac{1}{2} \Big) \Big] \Big] ... \Big[\sum_{n_{\boldsymbol{p}_N}=0}^{\infty} exp\Big[-\beta E_{\boldsymbol{p}_N} \Big(n_{\boldsymbol{p}_N} + \frac{1}{2} \Big) \Big] \Big] \\
&= \prod_{\boldsymbol{p}=-\infty}^{\infty} \Big[\sum_{n_{\boldsymbol{p}}=0}^{\infty} exp\Big[-\beta E_{\boldsymbol{p}} \Big(n_{\boldsymbol{p}} + \frac{1}{2} \Big) \Big] \Big].
\end{aligned} \tag{195}$$

This equation can be even more simplified. We can observe that $e^{-\frac{\beta E_{\boldsymbol{p}}}{2}}$ can be taking out of the summation and then the term left can be identified with

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \tag{196}$$

where $x < 1$. This identity can be found at chapter 5 of Ref. [2]. As the exponent of $e^{-\beta E_{\boldsymbol{p}}}$ contains only positive terms, it satisfies the condition to be less than zero. And so we finally have

$$\mathcal{Z} = \prod_{\boldsymbol{p}} \Big[\frac{e^{-\frac{\beta E_{\boldsymbol{p}}}{2}}}{1 - e^{-\beta E_{\boldsymbol{p}}}} \Big] = exp\left(-\frac{\beta}{2} \sum_{\boldsymbol{p}} E_{\boldsymbol{p}} \right) \prod_{\boldsymbol{p}} \Big[\frac{1}{1 - e^{-\beta E_{\boldsymbol{p}}}} \Big]. \tag{197}$$

This is the final form of the partition function with which further calculations will be done. We can observe that the first term is, again, divergent. And we can again call it the vacuum term. If we want, we can simplify the above equation even more as

$$\begin{aligned}
\mathcal{Z} &= \prod_{\boldsymbol{p}} \Big[\frac{e^{-\frac{\beta E_{\boldsymbol{p}}}{2}}}{1 - e^{-\beta E_{\boldsymbol{p}}}} \Big] = \prod_{\boldsymbol{p}} \Big[\frac{1}{e^{\frac{\beta E_{\boldsymbol{p}}}{2}} - e^{-\frac{\beta E_{\boldsymbol{p}}}{2}}} \Big] \\
&= 2 \prod_{\boldsymbol{p}} \Big[sinh\Big(\frac{\beta E_{\boldsymbol{p}}}{2} \Big) \Big]^{-1}.
\end{aligned} \tag{198}$$

D. Thermodynamic Potential

The thermodynamic potential is defined as in Eq.(10.78) from Ref.[4] as

$$\Omega = -\frac{1}{\beta}\ln\mathcal{Z}. \tag{199}$$

Let us calculate the Natural logarithm of the partition function. From eq. (197)

$$\begin{aligned}\ln\mathcal{Z} &= \ln\left\{exp\left(-\frac{\beta}{2}\sum_{\boldsymbol{p}}E_{\boldsymbol{p}}\right) + \ln\left[\prod_{\boldsymbol{p}}\left(\frac{1}{1-e^{-\beta E_{\boldsymbol{p}}}}\right)\right]\right.\\ &= -\frac{\beta}{2}\sum_{\boldsymbol{p}}E_{\boldsymbol{p}} + \sum_{\boldsymbol{p}}\ln\left(\frac{1}{1-e^{-\beta E_{\boldsymbol{p}}}}\right)\\ &= -\frac{\beta}{2}\sum_{\boldsymbol{p}}E_{\boldsymbol{p}} - \sum_{\boldsymbol{p}}\ln\left(1-e^{-\beta E_{\boldsymbol{p}}}\right)\end{aligned} \tag{200}$$

and so we have

$$\Omega = \frac{1}{2}\sum_{\boldsymbol{p}}E_{\boldsymbol{p}} + \frac{1}{\beta}\sum_{\boldsymbol{p}}\ln\left(1-e^{-\beta E_{\boldsymbol{p}}}\right). \tag{201}$$

E. Main Thermodynamic Quantities

In this section we want to calculate the averages of the number of the particles, energy and momentum as well as the pressure of the system. In order to do that we first define the statistical operator $\hat{\rho}$ [6]:

$$\hat{\rho} = \frac{1}{\mathcal{Z}}e^{-\beta\hat{H}}, \tag{202}$$

where was considered that the chemical potential μ is equal to zero.

The average of any operator $\hat{A}$ can be calculated using the following identity [4]:

$$< A >= Tr(\hat{\rho}\hat{A}). \tag{203}$$

So, if we want to calculate the average of the number of particles $n_{\boldsymbol{p}}$ we have

$$\begin{aligned}< n_{\boldsymbol{p}} > &= Tr(\hat{\rho}\hat{N}_{\boldsymbol{p}}) = Tr(\hat{\rho}\hat{a}_{\boldsymbol{p}}\hat{a}^{\dagger}_{\boldsymbol{p}})\\ &= \frac{1}{\mathcal{Z}}\sum_{n_{\boldsymbol{p}_1}}\sum_{n_{\boldsymbol{p}_2}}...\sum_{n_{\boldsymbol{p}_N}}\left\langle n_{\boldsymbol{p}_1}n_{\boldsymbol{p}_2}...n_{\boldsymbol{p}_N}\right| e^{-\beta\hat{H}}\hat{N}_{\boldsymbol{p}}\left|n_{\boldsymbol{p}_1}n_{\boldsymbol{p}_2}...n_{\boldsymbol{p}_N}\right\rangle\\ &= \frac{1}{\mathcal{Z}}\sum_{n_{\boldsymbol{p}_1}}\sum_{n_{\boldsymbol{p}_2}}...\sum_{n_{\boldsymbol{p}_N}}n_{\boldsymbol{p}}\left\langle n_{\boldsymbol{p}_1}n_{\boldsymbol{p}_2}...n_{\boldsymbol{p}_N}\right| e^{-\beta\hat{H}}\left|n_{\boldsymbol{p}_1}n_{\boldsymbol{p}_2}...n_{\boldsymbol{p}_N}\right\rangle\\ &= \frac{1}{\mathcal{Z}}\sum_{n_{\boldsymbol{p}_1}}\sum_{n_{\boldsymbol{p}_2}}...\sum_{n_{\boldsymbol{p}_N}}n_{\boldsymbol{p}}exp\Big[-\beta\sum_{\boldsymbol{p}}E_{\boldsymbol{p}}\Big(n_{\boldsymbol{p}}+\frac{1}{2}\Big)\Big].\end{aligned} \tag{204}$$

At the last line was used the result obtained at eq. (194). We observe that a exponential of a summation is a product of exponentials. This property was already used at eq. (195). With that we have

$$
\begin{aligned}
< n_{\boldsymbol{p}} > &= \frac{1}{\mathcal{Z}} \sum_{n_{\boldsymbol{p}_1}} exp\Big[-\beta E_{\boldsymbol{p}_1}\Big(n_{\boldsymbol{p}_1}+\frac{1}{2}\Big)\Big] \sum_{n_{\boldsymbol{p}_2}} exp\Big[-\beta E_{\boldsymbol{p}_2}\Big(n_{\boldsymbol{p}_2}+\frac{1}{2}\Big)\Big] ... \\
&\quad ... \sum_{n_{\boldsymbol{p}}} n_{\boldsymbol{p}} exp\Big[-\beta E_{\boldsymbol{p}}\Big(n_{\boldsymbol{p}}+\frac{1}{2}\Big)\Big] ... \sum_{n_{\boldsymbol{p}_N}} exp\Big[-\beta E_{\boldsymbol{p}_N}\Big(n_{\boldsymbol{p}_N}+\frac{1}{2}\Big)\Big] \\
&= \frac{\sum_{n_{\boldsymbol{p}}} n_{\boldsymbol{p}} exp\Big[-\beta E_{\boldsymbol{p}}\Big(n_{\boldsymbol{p}}+\frac{1}{2}\Big)\Big]}{\sum_{n_{\boldsymbol{p}}} exp\Big[-\beta E_{\boldsymbol{p}}\Big(n_{\boldsymbol{p}}+\frac{1}{2}\Big)\Big]} = \frac{\sum_{n_{\boldsymbol{p}}} n_{\boldsymbol{p}} exp\Big(-\beta E_{\boldsymbol{p}} n_{\boldsymbol{p}}\Big)}{\sum_{n_{\boldsymbol{p}}} exp\Big(-\beta E_{\boldsymbol{p}} n_{\boldsymbol{p}}\Big)}. \qquad (205)
\end{aligned}
$$

At the last line was used eq. (195). The term at the denominator can be simplified using eq. (196), but for the numerator we need

$$
x\frac{d}{dx}\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} n x^n = \frac{x}{(1-x)^2} \qquad (206)
$$

and so we are left with

$$
\begin{aligned}
< n_{\boldsymbol{p}} > &= \frac{\sum_{n_{\boldsymbol{p}}} n_{\boldsymbol{p}} exp\Big(-\beta E_{\boldsymbol{p}} n_{\boldsymbol{p}}\Big)}{\sum_{n_{\boldsymbol{p}}} exp\Big(-\beta E_{\boldsymbol{p}} n_{\boldsymbol{p}}\Big)} \\
&= \frac{e^{-\beta E_{\boldsymbol{p}}}}{(1-e^{-\beta E_{\boldsymbol{p}}})^2} \frac{(1-e^{-\beta E_{\boldsymbol{p}}})}{1} \\
&= \frac{e^{-\beta E_{\boldsymbol{p}}}}{1-e^{-\beta E_{\boldsymbol{p}}}}. \qquad (207)
\end{aligned}
$$

Finally we have

$$
< n_{\boldsymbol{p}} >= \frac{1}{e^{\beta E_{\boldsymbol{p}}}-1}. \qquad (208)
$$

In a similar way we can also calculate the average energy. For that we use the Hamiltonian given by eq. (165):

$$
\begin{aligned}
< H >= Tr(\hat{\rho}\hat{H}) &= \frac{1}{\mathcal{Z}} \sum_{n_{\boldsymbol{p}_1}} \sum_{n_{\boldsymbol{p}_2}} ... \sum_{n_{\boldsymbol{p}_N}} \langle n_{\boldsymbol{p}_1} n_{\boldsymbol{p}_2} ... n_{\boldsymbol{p}_N} | \, e^{-\beta\hat{H}} \hat{H} \, | n_{\boldsymbol{p}_1} n_{\boldsymbol{p}_2} ... n_{\boldsymbol{p}_N} \rangle \\
&= \frac{1}{\mathcal{Z}} \sum_{n_{\boldsymbol{p}_1}} \sum_{n_{\boldsymbol{p}_2}} ... \sum_{n_{\boldsymbol{p}_N}} \langle n_{\boldsymbol{p}_1} n_{\boldsymbol{p}_2} ... n_{\boldsymbol{p}_N} | \, e^{-\beta\hat{H}} \sum_{\boldsymbol{p}} E_{\boldsymbol{p}}\Big(\hat{N}_{\boldsymbol{p}}+\frac{1}{2}\Big) | n_{\boldsymbol{p}_1} n_{\boldsymbol{p}_2} ... n_{\boldsymbol{p}_N} \rangle \\
&= \frac{1}{\mathcal{Z}} \sum_{n_{\boldsymbol{p}_1}} \sum_{n_{\boldsymbol{p}_2}} ... \sum_{n_{\boldsymbol{p}_N}} \sum_{\boldsymbol{p}} E_{\boldsymbol{p}}\Big(n_{\boldsymbol{p}}+\frac{1}{2}\Big) \langle n_{\boldsymbol{p}_1} n_{\boldsymbol{p}_2} ... n_{\boldsymbol{p}_N} | \, e^{-\beta\hat{H}} \, | n_{\boldsymbol{p}_1} n_{\boldsymbol{p}_2} ... n_{\boldsymbol{p}_N} \rangle \\
&= \frac{1}{\mathcal{Z}} \sum_{n_{\boldsymbol{p}_1}} \sum_{n_{\boldsymbol{p}_2}} ... \sum_{n_{\boldsymbol{p}_N}} \sum_{\boldsymbol{p}} E_{\boldsymbol{p}}\Big(n_{\boldsymbol{p}}+\frac{1}{2}\Big) exp\Big[-\beta \sum_{\boldsymbol{p}} E_{\boldsymbol{p}'}\Big(n_{\boldsymbol{p}'}+\frac{1}{2}\Big)\Big]. \qquad (209)
\end{aligned}
$$

This equation can be separated into two equations

$$
\begin{aligned}
< H > &= \frac{1}{\mathcal{Z}} \sum_{n_{\boldsymbol{p}_1}} \sum_{n_{\boldsymbol{p}_2}} ... \sum_{n_{\boldsymbol{p}_N}} \sum_{\boldsymbol{p}} E_{\boldsymbol{p}} n_{\boldsymbol{p}} exp\Big[-\beta \sum_{\boldsymbol{p}} E_{\boldsymbol{p}'}\Big(n_{\boldsymbol{p}'}+\frac{1}{2}\Big)\Big] \\
&\quad + \frac{1}{\mathcal{Z}} \sum_{n_{\boldsymbol{p}_1}} \sum_{n_{\boldsymbol{p}_2}} ... \sum_{n_{\boldsymbol{p}_N}} \sum_{\boldsymbol{p}} \frac{E_{\boldsymbol{p}}}{2} exp\Big[-\beta \sum_{\boldsymbol{p}} E_{\boldsymbol{p}'}\Big(n_{\boldsymbol{p}'}+\frac{1}{2}\Big)\Big]. \qquad (210)
\end{aligned}
$$

Similar as was done in the calculations for the average number of particles, here we try to identify what is equal on the numerator and denominator. For the second term it is easy to see that the numerator and denominator are almost the same, only the summation over $\boldsymbol{p}$ that is different. For the first term we have a situation similar as at eq. (205)

$$\begin{aligned}
< H > &= \sum_{\boldsymbol{p}} E_{\boldsymbol{p}} \frac{\sum_{n_{\boldsymbol{p}}} n_{\boldsymbol{p}} exp\Big(- \beta E_{\boldsymbol{p}} n_{\boldsymbol{p}} \Big)}{\sum_{n_{\boldsymbol{p}}} exp\Big(- \beta E_{\boldsymbol{p}} n_{\boldsymbol{p}} \Big)} + \sum_{\boldsymbol{p}} \frac{E_{\boldsymbol{p}}}{2} \\
&= \sum_{\boldsymbol{p}} E_{\boldsymbol{p}} \frac{1}{(e^{\beta E_{\boldsymbol{p}}} - 1)} + \sum_{\boldsymbol{p}} \frac{E_{\boldsymbol{p}}}{2}
\end{aligned} \tag{211}$$

and we have our final result for the average energy

$$< H >= \sum_{\boldsymbol{p}} \frac{E_{\boldsymbol{p}}}{2} + \sum_{\boldsymbol{p}} \frac{E_{\boldsymbol{p}}}{(e^{\beta E_{\boldsymbol{p}}} - 1)}. \tag{212}$$

For the calculations of the average momentum we have to use eq. (164). As can be observed, this equation is not very different from the one of the Hamiltonian, therefore, the calculations here are very much same

$$\begin{aligned}
< \boldsymbol{P} > = Tr(\hat{\rho}\hat{\boldsymbol{P}}) &= \frac{1}{\mathcal{Z}} \sum_{n_{\boldsymbol{p}_1}} \sum_{n_{\boldsymbol{p}_2}} ... \sum_{n_{\boldsymbol{p}_N}} \langle n_{\boldsymbol{p}_1} n_{\boldsymbol{p}_2} ... n_{\boldsymbol{p}_N} \big| \, e^{-\beta \hat{H}} \hat{\boldsymbol{P}} \, \big| n_{\boldsymbol{p}_1} n_{\boldsymbol{p}_2} ... n_{\boldsymbol{p}_N} \rangle \\
&= \frac{1}{\mathcal{Z}} \sum_{n_{\boldsymbol{p}_1}} \sum_{n_{\boldsymbol{p}_2}} ... \sum_{n_{\boldsymbol{p}_N}} \langle n_{\boldsymbol{p}_1} n_{\boldsymbol{p}_2} ... n_{\boldsymbol{p}_N} \big| \, e^{-\beta \hat{H}} \sum_{\boldsymbol{p}} \boldsymbol{p} \Big(\hat{N}_{\boldsymbol{p}} + \frac{1}{2} \Big) \, \big| n_{\boldsymbol{p}_1} n_{\boldsymbol{p}_2} ... n_{\boldsymbol{p}_N} \rangle \\
&= \frac{1}{\mathcal{Z}} \sum_{n_{\boldsymbol{p}_1}} \sum_{n_{\boldsymbol{p}_2}} ... \sum_{n_{\boldsymbol{p}_N}} \sum_{\boldsymbol{p}} \boldsymbol{p} \Big(n_{\boldsymbol{p}} + \frac{1}{2} \Big) exp\Big[- \beta \sum_{\boldsymbol{p}} E_{\boldsymbol{p}'} \Big(n_{\boldsymbol{p}'} + \frac{1}{2} \Big) \Big] \\
&= \sum_{\boldsymbol{p}} \boldsymbol{p} \frac{1}{(e^{\beta E_{\boldsymbol{p}}} - 1)} + \sum_{\boldsymbol{p}} \frac{\boldsymbol{p}}{2}
\end{aligned} \tag{213}$$

and we have the finally form of the average momentum

$$< \boldsymbol{P} >= \sum_{\boldsymbol{p}} \frac{\boldsymbol{p}}{2} + \sum_{\boldsymbol{p}} \boldsymbol{p} \frac{1}{(e^{\beta E_{\boldsymbol{p}}} - 1)}. \tag{214}$$

The pressure can be calculated through the following relation

$$P = -Tr \Big(\hat{\rho} \frac{\partial \hat{H}}{\partial V} \Big), \tag{215}$$

where the chemical potential $\mu = 0$. The Hamiltonian is the one given by eq. (165) and we have to remember that the energy $E_{\boldsymbol{p}}$ is given by $E_{\boldsymbol{p}} = \sqrt{\boldsymbol{p}^2 + m^2}$ where $\boldsymbol{p} = 2\pi \boldsymbol{k}/L$ and $L = V^{1/3}$. With that, the derivation of the Hamiltonian is going to be

$$\frac{\partial \hat{H}}{\partial V} = \sum_{\boldsymbol{p}} \left(\frac{\partial E_{\boldsymbol{p}}}{\partial V} \right) \left(N_{\boldsymbol{p}} + \frac{1}{2} \right). \tag{216}$$

Let us calculate the derivation of the energy

$$\frac{\partial E_{\boldsymbol{p}}}{\partial V} = \frac{\partial}{\partial V}\sqrt{\left(\frac{2\pi \boldsymbol{k}}{V^{1/3}}\right)^2 + m^2}$$
$$= \frac{1}{2}\left[\left(\frac{2\pi \boldsymbol{k}}{V^{1/3}}\right)^2 + m^2\right]^{-1/2}\left(-\frac{2}{3}\right)V^{-2/3}(2\pi\boldsymbol{k})^2$$
$$= -\frac{1}{3}\frac{\boldsymbol{p}^2}{VE_{\boldsymbol{p}}}. \tag{217}$$

Replacing this result into eq.(216), we have

$$\frac{\partial \hat{H}}{\partial V} = -\frac{1}{3V}\sum_{\boldsymbol{p}}\frac{\boldsymbol{p}^2}{E_{\boldsymbol{p}}}\Big(N_{\boldsymbol{p}} + \frac{1}{2}\Big). \tag{218}$$

Comparing this result with the Hamiltonian that was used to do the calculations of the average energy and to the equation of the momentum we can see that the equation above is very similar, resulting on very similar calculations for the pressure

$$P = -Tr\Big(\hat{\rho}\frac{\partial \hat{H}}{\partial V}\Big) = -\frac{1}{\mathcal{Z}}\sum_{n_{\boldsymbol{p}_1}}\sum_{n_{\boldsymbol{p}_2}}...\sum_{n_{\boldsymbol{p}_N}}\langle n_{\boldsymbol{p}_1}n_{\boldsymbol{p}_2}...n_{\boldsymbol{p}_N}|\, e^{-\beta\hat{H}}\left(-\frac{1}{3V}\right)\sum_{\boldsymbol{p}}\frac{\boldsymbol{p}^2}{E_{\boldsymbol{p}}}\Big(N_{\boldsymbol{p}} + \frac{1}{2}\Big)\,|n_{\boldsymbol{p}_1}n_{\boldsymbol{p}_2}...n_{\boldsymbol{p}_N}\rangle$$
$$= \frac{1}{\mathcal{Z}}\frac{1}{3V}\sum_{n_{\boldsymbol{p}_1}}\sum_{n_{\boldsymbol{p}_2}}...\sum_{n_{\boldsymbol{p}_N}}\sum_{\boldsymbol{p}}\frac{\boldsymbol{p}^2}{E_{\boldsymbol{p}}}\Big(n_{\boldsymbol{p}} + \frac{1}{2}\Big)exp\Big[-\beta\sum_{\boldsymbol{p}}E_{\boldsymbol{p'}}\Big(n_{\boldsymbol{p'}} + \frac{1}{2}\Big)\Big]$$
$$= \frac{1}{3V}\sum_{\boldsymbol{p}}\frac{\boldsymbol{p}^2}{E_{\boldsymbol{p}}}\frac{1}{(e^{\beta E_{\boldsymbol{p}}} - 1)} + \frac{1}{3V}\sum_{\boldsymbol{p}}\frac{\boldsymbol{p}^2}{2E_{\boldsymbol{p}}}. \tag{219}$$

And we finally have the result for the pressure

$$P = \frac{1}{3V}\sum_{\boldsymbol{p}}\frac{\boldsymbol{p}^2}{2E_{\boldsymbol{p}}} + \frac{1}{3V}\sum_{\boldsymbol{p}}\frac{\boldsymbol{p}^2}{E_{\boldsymbol{p}}}\frac{1}{(e^{\beta E_{\boldsymbol{p}}} - 1)}. \tag{220}$$

V. PATH INTEGRAL

In this section we obtain analytically the partition function on the lattice in the configuration space in one spatial dimension by the path integral method and than, using the discrete Fourier transform, the partition function in the momentum space is obtained. The generalization to the three spatial dimension is done. Finally, the main thermodynamic results are obtained and compared with the results obtained on the previous section. The caculations made here follow the deriavations given in the paper [6].

A. Configuration space

The partition function given by eq. (183) can be rewritten in the path integral method as

$$\mathcal{Z} = \int [d\phi]\,\langle\phi|\, e^{\beta\hat{H}}\,|\phi\rangle\,, \tag{221}$$

where the Hamiltonian operator $\hat{H}$ is given by

$$\hat{H} = \int d^3\boldsymbol{x}\hat{\mathcal{H}}\left(\hat{\pi}(\boldsymbol{x},0),\hat{\phi}(\boldsymbol{x},0)\right) \tag{222}$$

and the Hamiltonian density is as in eq. (85)

$$\hat{\mathcal{H}}\left(\hat{\pi}(\boldsymbol{x},0),\hat{\phi}(\boldsymbol{x},0)\right) = \frac{1}{2}\left[\hat{\pi}^2(\boldsymbol{x},0) + (\boldsymbol{\nabla}\hat{\phi}(\boldsymbol{x},0))^2 + m^2\hat{\phi}^2(\boldsymbol{x},0)\right]. \tag{223}$$

All the operators are defined in the Schrödinger representation and do not depend on time t. From now on we introduce a shorter notation $\hat{\phi}(\boldsymbol{x},0) \equiv \hat{\phi}(\boldsymbol{x})$ and $\hat{\pi}(\boldsymbol{x},0) \equiv \hat{\pi}(\boldsymbol{x})$. In this picture we also have the following relations

$$\begin{aligned}\hat{\phi}(\boldsymbol{x})\left|\hat{\phi}(\boldsymbol{x})\right\rangle &= \phi(\boldsymbol{x})\left|\hat{\phi}(\boldsymbol{x})\right\rangle \\ \hat{\pi}(\boldsymbol{x})\left|\hat{\pi}(\boldsymbol{x})\right\rangle &= \pi(\boldsymbol{x})\left|\hat{\pi}(\boldsymbol{x})\right\rangle.\end{aligned} \tag{224}$$

In order to use the path integral method we need to dicretize β and the volume V. For that we divide β into N_β intervals of length a_β, so that $\beta = a_\beta N_\beta$ and the volume V of the system is divided into N_σ^3 small cells, each with volume a_σ^3 so that $V = L_\sigma^3$ and $L_\sigma = a_\sigma N_\sigma$. The small intervals on the β-axis are labeled by the integer $n_\beta = 1, ..., N_\beta$ and the space cells in V are fixed by the integer vector $\boldsymbol{n} = (n_x, n_y, n_z)$ with the coordinates $n_\alpha = 1, ..., N_\sigma$. The discretized β and volume V form a 4-dimensional lattice Γ the cells of which are given by the vector $n_\nu = (\boldsymbol{n}, n_\beta)$. So we have that for each cell n_ν there is a field operator $\hat{\phi}(n_\nu)$ with an eigenstate $|\phi(n_\nu)\rangle$ and eigenvalue $\phi(n_\nu)$. From now on we use another notation $\hat{\phi}_{l,i} \equiv \hat{\phi}(n_\nu)$ ($l \in \Lambda_3, i = n_\beta$), where l is an integer and Λ_3 is the 3-dimensional subspace of the lattice Λ.

The partition function can be written as

$$\mathcal{Z} = \lim_{N_\sigma, N_\beta \to \infty} \mathcal{Z}_{lat}, \tag{225}$$

where $\mathcal{Z}_{lat}$ for the 3-dimensional space is given by

$$\mathcal{Z}_{lat} = \int \left(\prod_{\boldsymbol{n}\in\Lambda_3} d\phi_{\boldsymbol{n},1}\right) \langle \phi_{1,1}, ..., \phi_{N_\sigma^3,1}| \, e^{-a_\beta \hat{H}} ... e^{-a_\beta \hat{H}} \, |\phi_{1,1}, ..., \phi_{N_\sigma^3,1}\rangle. \tag{226}$$

It is easier to calculate the $\mathcal{Z}_{lat}$ in one spatial dimension. In that case we have

$$\mathcal{Z}_{lat} = \int \left(\prod_{l=1}^{N_\sigma} d\phi_{l,1}\right) \langle \phi_{1,1}, ..., \phi_{N_\sigma,1}| \, e^{-a_\beta \hat{H}} ... e^{-a_\beta \hat{H}} \, |\phi_{1,1}, ..., \phi_{N_\sigma,1}\rangle. \tag{227}$$

The state vectors are orthogonal

$$\langle \phi_i | \phi_j \rangle = \langle \phi_{1,i}, ..., \phi_{N_\sigma,i} | \phi_{1,i}, ..., \phi_{N_\sigma,j} \rangle = \prod_{l=1}^{N_\sigma} \delta(\phi_{1,i} - \phi_{1,j}), \tag{228}$$

$$\langle \pi_i | \pi_j \rangle = \frac{2\pi}{a_\sigma} \langle \pi_{1,i}, ..., \pi_{N_\sigma,i} | \pi_{1,i}, ..., \pi_{N_\sigma,j} \rangle = \prod_{l=1}^{N_\sigma} \frac{2\pi}{a_\sigma} \delta(\phi_{1,i} - \phi_{1,j}) \tag{229}$$

and complete

$$1 = \int \prod_{l=1}^{N_\sigma} d\phi_{1,i} \, |\phi_i\rangle \langle \phi_i| = \int \prod_{l=1}^{N_\sigma} d\phi_{1,i} \, |\phi_{1,i}, ..., \phi_{N_\sigma,i}\rangle \langle \phi_{1,i}, ..., \phi_{N_\sigma,i}|, \tag{230}$$

$$1 = \int \prod_{l=1}^{N_\sigma} \frac{a_\sigma}{2\pi} d\pi_{1,i} |\pi_i\rangle \langle\pi_i| = \int \prod_{l=1}^{N_\sigma} \frac{a_\sigma}{2\pi} d\pi_{1,i} |\pi_{1,i}, ..., \pi_{N_\sigma,i}\rangle \langle\pi_{1,i}, ..., \pi_{N_\sigma,i}| \,. \tag{231}$$

To solve eq. (227) we insert in the left side of each i-th exponent the product of two unit operators for the fields $|\phi_{i+1}\rangle$ and $|\pi_i\rangle$

$$\begin{aligned}
\mathcal{Z}_{lat} = & \int \left(\prod_{l=1}^{N_\sigma} d\phi_{l,1}\right) \int \left(\prod_{i=1}^{N_\beta}\prod_{l=1}^{N_\sigma} \frac{a_\sigma d\pi_{l,i} d\phi_{l,i+1}}{2\pi}\right) \langle\phi_1|\phi_{N_\beta+1}\rangle \langle\phi_{N_\beta+1}|\pi_{N_\beta}\rangle \langle\pi_{N_\beta}| e^{-a_\beta \hat{H}} |\phi_{N_\beta}\rangle \\
& \langle\phi_{N_\beta}|\pi_{N_\beta-1}\rangle \langle\pi_{N_\beta-1}| e^{-a_\beta \hat{H}} |\phi_{N_\beta-1}\rangle ... \langle\phi_3|\pi_2\rangle e^{-a_\beta \hat{H}} |\phi_2\rangle \langle\phi_2|\pi_1\rangle \langle\pi_1| e^{-a_\beta \hat{H}} |\phi_1\rangle \\
= & \int \left(\prod_{l=1}^{N_\sigma} d\phi_{l,1}\right) \int \left(\prod_{i=1}^{N_\beta}\prod_{l=1}^{N_\sigma} \frac{a_\sigma d\pi_{l,i} d\phi_{l,i+1}}{2\pi}\right) \langle\phi_1|\phi_{N_\beta+1}\rangle \prod_{i=1}^{N_\beta} \langle\phi_{i+1}|\pi_i\rangle \langle\pi_i| e^{-a_\beta \hat{H}} |\phi_i\rangle \,.
\end{aligned} \tag{232}$$

We can now calculate the matrix elements $\langle\pi_i| e^{-a_\beta \hat{H}} |\phi_i\rangle$. For that we make a expansion of the exponential

$$\langle\pi_i| e^{-a_\beta \hat{H}} |\phi_i\rangle = \langle\pi_i| 1 - \frac{a_\beta \hat{H}}{1!} + \frac{a_\beta^2 \hat{H}^2}{2!} + ... |\phi_i\rangle \tag{233}$$

and use the Hamiltonian operator given by eq. (223) in the discrete form. For only one spatial dimension the identity $\left(\int dx \leftrightarrow a_\sigma \sum_{l=1}^{N_\sigma}\right)$ can be used and we have to remember the definition of a derivation as follows

$$\frac{\partial\hat{\phi}(x)}{\partial x} = \frac{\hat{\phi}(x+\Delta x) - \hat{\phi}(x)}{\Delta x} = \frac{\hat{\phi}_{l+1,i} - \hat{\phi}_{l,i}}{a_\sigma} \tag{234}$$

and so we have our Hamiltonian operator in the discrete form

$$\hat{H} = \frac{1}{2} a_\sigma \sum_{l=1}^{N_\sigma} \left[\hat{\pi}_{l,i}^2 + \left(\frac{\hat{\phi}_{l+1,i} - \hat{\phi}_{l,i}}{a_\sigma}\right)^2 + m^2 \hat{\phi}_{l,i}^2\right] . \tag{235}$$

Let us take a look at the second term of the expansion in eq. (233). The matrix element

$$\begin{aligned}
\langle\pi_i| \hat{H} |\phi_i\rangle &= \langle\pi_i| \frac{1}{2} a_\sigma \sum_{l=1}^{N_\sigma} \left[\hat{\pi}_{l,i}^2 + \left(\frac{\hat{\phi}_{l+1,i} - \hat{\phi}_{l,i}}{a_\sigma}\right)^2 + m^2 \hat{\phi}_{l,i}^2\right] |\phi_i\rangle \\
&= \frac{1}{2} a_\sigma \sum_{l=1}^{N_\sigma} \left[\pi_{l,i}^2 + \left(\frac{\phi_{l+1,i} - \phi_{l,i}}{a_\sigma}\right)^2 + m^2 \phi_{l,i}^2\right] \langle\pi_i|\phi_i\rangle \,,
\end{aligned} \tag{236}$$

where we used eq. (224). Replacing this into eq. (233), we have

$$\begin{aligned}
\langle\pi_i| e^{-a_\beta \hat{H}} |\phi_i\rangle &= \langle\pi_i|\phi_i\rangle \left[1 - \frac{a_\sigma a_\beta}{2} \sum_{l=1}^{N_\sigma} \left(\pi_{l,i}^2 + \left(\frac{\phi_{l+1,i} - \phi_{l,i}}{a_\sigma}\right)^2 + m^2 \phi_{l,i}^2\right) + ...\right] \\
&= \langle\pi_i|\phi_i\rangle \, e^{-\frac{a_\sigma a_\beta}{2} \sum_{l=1}^{N_\sigma} \left(\pi_{l,i}^2 + \left(\frac{\phi_{l+1,i} - \phi_{l,i}}{a_\sigma}\right)^2 + m^2 \phi_{l,i}^2\right)} .
\end{aligned} \tag{237}$$

We also have

$$\langle \pi_i | \phi_i \rangle = e^{-ia_\sigma \sum_{l=1}^{N_\sigma} \pi_{l,i}\phi_{l,i}}$$
$$\langle \phi_{i+1} | \pi_i \rangle = e^{ia_\sigma \sum_{l=1}^{N_\sigma} \pi_{l,i}\phi_{l,i+1}}. \tag{238}$$

Substituting eqs. (237) and (238) into eq. (232), we obtain

$$\mathcal{Z}_{lat} = \int \left(\prod_{l=1}^{N_\sigma} d\phi_{l,1} \right) \int \left(\prod_{i=1}^{N_\beta} \prod_{l=1}^{N_\sigma} \frac{a_\sigma d\pi_{l,i} d\phi_{l,i+1}}{2\pi} \right) \langle \phi_1 | \phi_{N_\beta+1} \rangle$$
$$e^{-a_\sigma a_\beta \sum_{i=1}^{N_\beta} \sum_{l=1}^{N_\sigma} \left[\frac{1}{2}\pi_{l,i}^2 + \frac{1}{2}\left(\frac{\phi_{l+1,i}-\phi_{l,i}}{a_\sigma} \right)^2 + \frac{1}{2}m^2\phi_{l,i}^2 - i\pi_{l,i}\left(\frac{\phi_{l,i+1}-\phi_{l,i}}{a_\beta} \right) \right]}. \tag{239}$$

Now let us take a look at the integration over $\pi_{l,i}$. To solve it we can use the result

$$\int_{-\infty}^{\infty} dx e^{-\alpha x^2 + \beta x} = e^{\frac{\beta^2}{4\alpha}} \left(\frac{\pi}{\alpha} \right)^{1/2} \tag{240}$$

and identify $\alpha = \dfrac{a_\sigma a_\beta}{2}$ and $\beta = ia_\sigma a_\beta \left(\dfrac{\phi_{l,i+1} - \phi_{l,i}}{a_\beta} \right)$. With that we obtain

$$\int_{-\infty}^{\infty} \prod_{i=1}^{N_\beta} \prod_{l=1}^{N_\sigma} \frac{a_\sigma}{2\pi} d\pi_{l,i} e^{-a_\sigma a_\beta \sum_{i=1}^{N_\beta} \sum_{l=1}^{N_\sigma} \left[\frac{1}{2}\pi_{l,i}^2 - ia_\sigma a_\beta \left(\frac{\phi_{l,i+1}-\phi_{l,i}}{a_\beta} \right) \right]}$$
$$= \left(\frac{a_\sigma}{2\pi a_\beta} \right)^{\frac{N_\beta N_\sigma}{2}} e^{-\frac{a_\sigma a_\beta}{2} \sum_{i=1}^{N_\beta} \sum_{l=1}^{N_\sigma} \left(\frac{\phi_{l,i+1}-\phi_{l,i}}{a_\beta} \right)^2}$$
$$= \prod_{i=1}^{N_\beta} \prod_{l=1}^{N_\sigma} \left(\frac{a_\sigma}{2\pi a_\beta} \right)^{1/2} e^{-\frac{a_\sigma a_\beta}{2} \left(\frac{\phi_{l,i+1}-\phi_{l,i}}{a_\beta} \right)^2}. \tag{241}$$

Substituting this result into eq. (239) we have

$$\mathcal{Z}_{lat} = \left(\frac{a_\sigma}{2\pi a_\beta} \right)^{\frac{N_\beta N_\sigma}{2}} \int \left(\prod_{l=1}^{N_\sigma} d\phi_{l,1} \right) \left(\prod_{i=1}^{N_\beta} \prod_{l=1}^{N_\sigma} d\phi_{l,i+1} \right) \langle \phi_1 | \phi_{N_\beta+1} \rangle$$
$$e^{-a_\sigma a_\beta \sum_{i=1}^{N_\beta} \sum_{l=1}^{N_\sigma} \left[\frac{1}{2}\left(\frac{\phi_{l,i+1}-\phi_{l,i}}{a_\beta} \right)^2 + \frac{1}{2}\left(\frac{\phi_{l+1,i}-\phi_{l,i}}{a_\sigma} \right)^2 + \frac{m^2\phi_{l,i}^2}{2} \right]}. \tag{242}$$

Now we want to use eq. (228) to make some simplifications. For that, it is also interesting if we rewrite the products that appear in the equation above

$$\left(\prod_{l=1}^{N_\sigma} d\phi_{l,1} \right) \left(\prod_{i=1}^{N_\beta} \prod_{l=1}^{N_\sigma} d\phi_{l,i+1} \right) = \prod_{l=1}^{N_\sigma} d\phi_{l,1} d\phi_{l,2} d\phi_{l,3} ... d\phi_{l,N_\beta+1}$$
$$= \prod_{l=1}^{N_\sigma} \left(\prod_{i=1}^{N_\beta} d\phi_{l,i} \right) d\phi_{l,N_\beta+1}$$
$$= \left(\prod_{l=1}^{N_\sigma} \prod_{i=1}^{N_\beta} d\phi_{l,i} \right) \left(\prod_{l=1}^{N_\sigma} d\phi_{l,N_\beta+1} \right). \tag{243}$$

Replacing this into eq. (242) and using eq. (228), we have

$$\mathcal{Z}_{lat} = \left(\frac{a_\sigma}{2\pi a_\beta}\right)^{\frac{N_\beta N_\sigma}{2}} \int \left(\prod_{l=1}^{N_\sigma}\prod_{i=1}^{N_\beta} d\phi_{l,i}\right)\left(\prod_{l=1}^{N_\sigma} d\phi_{l,N_\beta+1}\right)\prod_{l=1}^{N_\sigma}\delta\left(\phi_{l,1}-\phi_{l,N_\beta+1}\right)$$
$$e^{-a_\sigma a_\beta \sum_{i=1}^{N_\beta}\sum_{l=1}^{N_\sigma}\left[\frac{1}{2}\left(\frac{\phi_{l,i+1}-\phi_{l,i}}{a_\beta}\right)^2+\frac{1}{2}\left(\frac{\phi_{l+1,i}-\phi_{l,i}}{a_\sigma}\right)^2+\frac{m^2\phi_{l,i}^2}{2}\right]}$$
$$= \left(\frac{a_\sigma}{2\pi a_\beta}\right)^{\frac{N_\beta N_\sigma}{2}} \left(\prod_{l=1}^{N_\sigma}\prod_{i=1}^{N_\beta} d\phi_{l,i}\right) e^{-a_\sigma a_\beta \sum_{i=1}^{N_\beta}\sum_{l=1}^{N_\sigma}\left[\frac{1}{2}\left(\frac{\phi_{l,i+1}-\phi_{l,i}}{a_\beta}\right)^2+\frac{1}{2}\left(\frac{\phi_{l+1,i}-\phi_{l,i}}{a_\sigma}\right)^2+\frac{m^2\phi_{l,i}^2}{2}\right]} \tag{244}$$

with the conditions that

$$\phi_{N_\sigma+1,i} = \phi_{1,i} \qquad \text{and} \qquad \phi_{l,N_\beta+1} = \phi_{l,1}. \tag{245}$$

Now our aim is to write the function in the exponent of eq. (244) into the bilinear form, which means that we want it into the $\boldsymbol{x}^T A\boldsymbol{y} = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_i y_j$ form, where A is a n x n matrix. Let us call the function in the exponent $D(\phi)$ and take a closer look at it:

$$D(\phi) = -a_\sigma a_\beta \sum_{i=1}^{N_\beta}\sum_{l=1}^{N_\sigma}\left[\frac{1}{2}\left(\frac{\phi_{l,i+1}-\phi_{l,i}}{a_\beta}\right)^2+\frac{1}{2}\left(\frac{\phi_{l+1,i}-\phi_{l,i}}{a_\sigma}\right)^2+\frac{m^2\phi_{l,i}^2}{2}\right]$$
$$= -a_\sigma a_\beta \sum_{i=1}^{N_\beta}\sum_{l=1}^{N_\sigma}\left[\frac{1}{2}\left(\frac{\phi_{l,i+1}^2+\phi_{l,i}^2-2\phi_{l,i+1}\phi_{l,i}}{a_\beta^2}\right)+\frac{1}{2}\left(\frac{\phi_{l+1,i}^2+\phi_{l,i}^2-2\phi_{l+1,i}\phi_{l,i}}{a_\sigma^2}\right)+\frac{m^2\phi_{l,i}^2}{2}\right]. \tag{246}$$

We can see that there are five types of summations in $D(\phi)$. Using the conditions (245) it is easy to verify that three of them are equal

$$\sum_{l=1}^{N_\sigma}\sum_{i=1}^{N_\beta}\phi_{l,i}^2 = \sum_{l=1}^{N_\sigma}\sum_{i=1}^{N_\beta}\phi_{l,i+1}^2 = \sum_{l=1}^{N_\sigma}\sum_{i=1}^{N_\beta}\phi_{l+1,i}^2$$
$$= \sum_{l=1}^{N_\sigma}\sum_{i=1}^{N_\beta}\sum_{k=1}^{N_\sigma}\sum_{j=1}^{N_\beta}\phi_{l,i}\delta_{k,l}\delta_{j,i}\phi_{k,j}, \tag{247}$$

where at the last line was used $\phi_{l,i} = \sum_{k=1}^{N_\sigma}\sum_{j=1}^{N_\beta}\delta_{k,l}\delta_{j,i}\phi_{k,j}$. The other two summations, being $\sum_{l=1}^{N_\sigma}\sum_{i=1}^{N_\beta}\phi_{l,i+1}\phi_{l,i}$ and $\sum_{l=1}^{N_\sigma}\sum_{i=1}^{N_\beta}\phi_{l+1,i}\phi_{l,i}$, need to be looked at a little bit closer

$$\sum_{l=1}^{N_\sigma}\sum_{i=1}^{N_\beta}\phi_{l,i+1}\phi_{l,i} = \sum_{l=1}^{N_\sigma}\sum_{i=1}^{N_\beta-1}\phi_{l,i+1}\phi_{l,i} + \sum_{l=1}^{N_\sigma}\phi_{l,N_\beta+1}\phi_{l,N_\beta} = \sum_{l=1}^{N_\sigma}\sum_{i=1}^{N_\beta}\left[(1-\delta_{i,N_\beta})\phi_{l,i+1}+\delta_{i,N_\beta}\phi_{l,1}\right]\phi_{l,i}$$
$$= \frac{1}{2}\sum_{l=1}^{N_\sigma}\sum_{i=1}^{N_\beta}\left[(1-\delta_{i,N_\beta})\phi_{l,i+1}+\delta_{i,N_\beta}\phi_{l,1}\right]\phi_{l,i} + \frac{1}{2}\sum_{k=1}^{N_\sigma}\sum_{j=1}^{N_\beta}\left[(1-\delta_{j,N_\beta})\phi_{k,j+1}+\delta_{j,N_\beta}\phi_{k,1}\right]\phi_{k,j}$$
$$= \frac{1}{2}\sum_{l=1}^{N_\sigma}\sum_{i=1}^{N_\beta}\sum_{k=1}^{N_\sigma}\sum_{j=1}^{N_\beta}\delta_{l,k}\left[(1-\delta_{i,N_\beta})\delta_{j,i+1}+\delta_{i,N_\beta}\delta_{j,1}+(1-\delta_{j,N_\beta})\delta_{i,j+1}+\delta_{j,N_\beta}\delta_{i,1}\right]\phi_{k,j}\phi_{l,i}, \tag{248}$$

where we used the same tactic as was used at the last line of eq. (247). Doing similar calculations for the summation left we end up with

$$\sum_{l=1}^{N_\sigma}\sum_{i=1}^{N_\beta}\phi_{l+1,i}\phi_{l,i} = \frac{1}{2}\sum_{l=1}^{N_\sigma}\sum_{i=1}^{N_\beta}\sum_{k=1}^{N_\sigma}\sum_{j=1}^{N_\beta}\delta_{i,j}\left[(1-\delta_{l,N_\sigma})\delta_{k,l+1}+\delta_{l,N_\sigma}\delta_{k,1}+(1-\delta_{k,N_\sigma})\delta_{l,k+1}+\delta_{k,N_\sigma}\delta_{l,1}\right]\phi_{k,j}\phi_{l,i}. \quad (249)$$

Now we can replace the results obtained on eqs. (247), (248) and (249) into eq. (244):

$$\begin{aligned}
\mathcal{Z}_{lat} &= \left(\frac{a_\sigma}{2\pi a_\beta}\right)^{\frac{N_\beta N_\sigma}{2}} \int \left(\prod_{l=1}^{N_\sigma}\prod_{i=1}^{N_\beta} d\phi_{l,i}\right) exp\Bigg\{ - a_\sigma a_\beta \left[\left(\frac{1}{a_\beta^2}+\frac{1}{a_\sigma^2}+\frac{m^2}{2}\right)\left(\sum_{l=1}^{N_\sigma}\sum_{i=1}^{N_\beta}\phi_{l,i}^2\right)\right.\\
&\left.-\sum_{l=1}^{N_\sigma}\sum_{i=1}^{N_\beta}\left(\frac{\phi_{l,i+1}\phi_{l,i}}{a_\beta^2}+\frac{\phi_{l+1,i}\phi_{l,i}}{a_\sigma^2}\right)\right]\Bigg\}\\
&= \left(\frac{a_\sigma}{2\pi a_\beta}\right)^{\frac{N_\beta N_\sigma}{2}} \left(\prod_{l=1}^{N_\sigma}\prod_{i=1}^{N_\beta} d\phi_{l,i}\right) exp\Bigg\{ - a_\sigma a_\beta \Bigg\{ \sum_{l=1}^{N_\sigma}\sum_{i=1}^{N_\beta}\sum_{k=1}^{N_\sigma}\sum_{j=1}^{N_\beta}\Bigg[\left(\frac{1}{a_\beta^2}+\frac{1}{a_\sigma^2}+\frac{m^2}{2}\right)\delta_{k,l}\delta_{j,i}\\
&-\frac{1}{2a_\beta^2}\left\{\delta_{l,k}\left[(1-\delta_{i,N_\beta})\delta_{j,i+1}+\delta_{i,N_\beta}\delta_{j,1}+(1-\delta_{j,N_\beta})\delta_{i,j+1}+\delta_{j,N_\beta}\delta_{i,1}\right]\right\}\\
&-\frac{1}{2a_\sigma^2}\left\{\delta_{i,j}\left[(1-\delta_{l,N_\sigma})\delta_{k,l+1}+\delta_{l,N_\sigma}\delta_{k,1}+(1-\delta_{k,N_\sigma})\delta_{l,k+1}+\delta_{k,N_\sigma}\delta_{l,1}\right]\phi_{k,j}\phi_{l,i}\right\}\Bigg]\phi_{k,j}\phi_{l,i}\Bigg\}\Bigg\}. \quad (250)
\end{aligned}$$

We can rewrite this expression in terms of $\phi_{l,i}^{'}$ where $\phi_{l,i}^{'} = \left(\frac{a_\sigma}{2a_\beta}\right)^{1/2}\phi_{l,i}$:

$$\begin{aligned}
\mathcal{Z}_{lat} &= \pi^{-\frac{N_\beta N_\sigma}{2}} \int \left(\prod_{l=1}^{N_\sigma}\prod_{i=1}^{N_\beta} d\phi_{l,i}^{'}\right) exp\Bigg\{ -\sum_{l=1}^{N_\sigma}\sum_{i=1}^{N_\beta}\sum_{k=1}^{N_\sigma}\sum_{j=1}^{N_\beta}\Bigg[\left(2\left(1+\frac{a_\beta^2}{a_\sigma^2}\right)+a_\beta^2 m^2\right)\delta_{k,l}\delta_{j,i}\\
&-\delta_{l,k}\left[(1-\delta_{i,N_\beta})\delta_{j,i+1}+\delta_{i,N_\beta}\delta_{j,1}+(1-\delta_{j,N_\beta})\delta_{i,j+1}+\delta_{j,N_\beta}\delta_{i,1}\right]\\
&-\frac{a_\beta^2}{a_\sigma^2}\left\{\delta_{i,j}\left[(1-\delta_{l,N_\sigma})\delta_{k,l+1}+\delta_{l,N_\sigma}\delta_{k,1}+(1-\delta_{k,N_\sigma})\delta_{l,k+1}+\delta_{k,N_\sigma}\delta_{l,1}\right]\phi_{k,j}\phi_{l,i}\right\}\Bigg]\phi_{k,j}^{'}\phi_{l,i}^{'}\Bigg\}. \quad (251)
\end{aligned}$$

And now we can see that the function in the exponent is in a bilinear form and we can identify $A'_{li,kj}$ as

$$\begin{aligned}
A'_{li,kj} &= \left(2\left(1+\frac{a_\beta^2}{a_\sigma^2}\right)+a_\beta^2 m^2\right)\delta_{k,l}\delta_{j,i} - \delta_{l,k}\left[(1-\delta_{i,N_\beta})\delta_{j,i+1}+\delta_{i,N_\beta}\delta_{j,1}+(1-\delta_{j,N_\beta})\delta_{i,j+1}+\delta_{j,N_\beta}\delta_{i,1}\right]\\
&-\frac{a_\beta^2}{a_\sigma^2}\delta_{i,j}\left[(1-\delta_{l,N_\sigma})\delta_{k,l+1}+\delta_{l,N_\sigma}\delta_{k,1}+(1-\delta_{k,N_\sigma})\delta_{l,k+1}+\delta_{k,N_\sigma}\delta_{l,1}\right] \quad (252)
\end{aligned}$$

resulting in

$$\mathcal{Z}_{lat} = \pi^{-\frac{N_\beta N_\sigma}{2}} \int \prod_{l=1}^{N_\sigma}\prod_{i=1}^{N_\beta} d\phi_{l,i}^{'} e^{-\sum_{l=1}^{N_\sigma}\sum_{i=1}^{N_\beta}\sum_{k=1}^{N_\sigma}\sum_{j=1}^{N_\beta} A'_{li,kj}\phi_{k,j}^{'}\phi_{l,i}^{'}}. \quad (253)$$

We can also substitute the coordinates by the number of the site on the lattice in the configuration space $J(l,i)$, where $J(l,i) = N_\beta(l-1)+i$. And so we finally have

$$\mathcal{Z}_{lat} = \pi^{-\frac{N_\beta N_\sigma}{2}} \int_{-\infty}^{\infty} \prod_{l=1}^{N_\sigma}\prod_{i=1}^{N_\beta} d\Phi_{J(l,i)} e^{-\sum_{l=1}^{N_\sigma}\sum_{i=1}^{N_\beta}\sum_{k=1}^{N_\sigma}\sum_{j=1}^{N_\beta} A_{J(l,i)J(k,j)}\Phi_{J(l,i)}\Phi_{J(k,j)}}. \quad (254)$$

To solve this integral we can use the following formula for Riemann integrals

$$\int_{-\infty}^{\infty} dx_1...dx_n e^{-x_i D_{ij} x_j} = \frac{\pi^{n/2}}{\sqrt{detD}}. \tag{255}$$

And so we have

$$\mathcal{Z}_{lat} = \frac{1}{\sqrt{detA}}. \tag{256}$$

Replacing this result into eq. (225), we have our partition function in the configuration space:

$$\mathcal{Z} = \lim_{N_\beta N_\sigma \to \infty} \frac{1}{\sqrt{detA}}. \tag{257}$$

B. Momentum space

In this section we want to rewrite the partition function in the momentum space on the basis of the Fourier transform defined on the lattice. Let us start with the calculations of the four-dimensional momentum.

The wave function in one-dimensional and continuous space is given by

$$u_{p_x}(x) = \frac{1}{\sqrt{L}} e^{ip_x x}. \tag{258}$$

If we confine this wave in a line of length L, we have the boundary conditions $u_{p_x}(0) = u_{p_x}(L)$, which gives us $e^{ip_x L} = 1$, resulting in

$$p_x = \frac{2\pi}{L} k_x \qquad \text{with} \qquad k_x = 0, \pm 1, \pm 2, ... \tag{259}$$

If we dicretize our x-axis, we have $x = a_\sigma l$, where $l = 1, ..., N_\sigma$, $L = a_\sigma N_\sigma$ and our wave function is

$$u_{p_x}(x) = \frac{1}{\sqrt{L}} e^{ip_x a_\sigma l}. \tag{260}$$

And in this case, our boundary conditions are $u_{p_x}(a_\sigma(N_\sigma + 1)) = u_{p_x}(a_\sigma)$, which gives us $e^{ip_x a_\sigma N_\sigma} = 1$, resulting in

$$p_x = \frac{2\pi}{a_\sigma N_\sigma} k_x \qquad \text{with} \qquad k_{x1} \leq k_x \leq k_{x2} \tag{261}$$

which can be generalized to our four-dimensional lattice

$$p_\mu = \frac{2\pi}{a_\mu N_\mu} k_\mu \qquad \text{with} \qquad \mu = \sigma, \beta \qquad \text{and} \qquad k_{\mu 1} \leq k_\mu \leq k_{\mu 2}, \tag{262}$$

where

$$k_{\mu 1} = -\frac{N_\mu - 1}{2} + \frac{\eta_\mu}{2}, \tag{263}$$

$$k_{\mu 1} = \frac{N_\mu - 1}{2} + \frac{\eta_\mu}{2}, \tag{264}$$

where we have $\eta_\mu = 1$ for N_μ even and $\eta_\mu = 0$ for N_μ odd.

It is interesting to note that even if the space is not discretized, we have an discretized momentum and, if the space is discretized, the number of momentums is equal to the number of intervals of the space. The vectors p_μ and n_μ satisfy some relations. Using the relation

$$\sum_{k=0}^{n} x^k = \frac{1-x^{n+1}}{1-x} \tag{265}$$

and adapting it to our case

$$\sum_{k=1}^{n} x^k = \frac{1-x^{n+1}}{1-x} - 1 = \frac{x(1-x^n)}{1-x} \qquad , \qquad x \neq 0 \tag{266}$$

we have

$$\frac{1}{N_\mu}\sum_{n_\mu=1}^{N_\mu} e^{\frac{2\pi i}{N_\mu}\left(k_\mu - k_\mu^{'}\right)n_\mu} = \frac{1}{N_\mu}\frac{e^{\frac{2\pi i}{N_\mu}\left(k_\mu - k_\mu^{'}\right)}\left(1 - e^{2\pi i\left(k_\mu - k_\mu^{'}\right)}\right)}{1 - e^{\frac{2\pi i}{N_\mu}\left(k_\mu - k_\mu^{'}\right)}} = \delta_{k_\mu, k_\mu^{'}} \tag{267}$$

and

$$\frac{1}{N_\mu}\sum_{k_\mu=k_{\mu 1}}^{k_\mu=k_{\mu 2}} e^{\frac{2\pi i}{N_\mu}\left(n_\mu - n_\mu^{'}\right)k_\mu} = 1 \qquad \text{if} \qquad n_\mu - n_\mu^{'} = 0. \tag{268}$$

In order to solve this summation for the case where $n_\mu - n_\mu^{'} \neq 0$ we have to have the summation start at $k_\mu = 1$. For that we can make the substitution $k_\mu^{'} = k_\mu - k_{\mu 1} + 1$, which gives us

$$\begin{aligned} &\frac{1}{N_\mu} e^{\frac{2\pi i}{N_\mu}\left(n_\mu - n_\mu^{'}\right)(k_{\mu 1}-1)} \sum_{k_\mu^{'}=1}^{N_\mu} e^{\frac{2\pi i}{N_\mu}\left(n_\mu - n_\mu^{'}\right)k_\mu^{'}} \\ &= \frac{1}{N_\mu} e^{\frac{2\pi i}{N_\mu}\left(n_\mu - n_\mu^{'}\right)(k_{\mu 1}-1)} \frac{1 - e^{2\pi i\left(n_\mu - n_\mu^{'}\right)}}{1 - e^{\frac{2\pi i}{N_\mu}\left(n_\mu - n_\mu^{'}\right)}} e^{\frac{2\pi i}{N_\mu}\left(n_\mu - n_\mu^{'}\right)} \\ &= 0 \qquad , \text{if} \qquad n_\mu - n_\mu^{'} \neq 0. \end{aligned} \tag{269}$$

Putting the two results together, we have the following relation for n_μ

$$\frac{1}{N_\mu}\sum_{k_\mu=k_{\mu 1}}^{k_\mu=k_{\mu 2}} e^{\frac{2\pi i}{N_\mu}\left(n_\mu - n_\mu^{'}\right)k_\mu} = \delta_{n_\mu, n_\mu^{'}}. \tag{270}$$

We also have

$$\frac{1}{N_\mu}\sum_{n_\mu=1}^{N_\mu} e^{i a_\mu\left(p_\mu - p_\mu^{'}\right)n_\mu} = \frac{1}{N_\mu}\sum_{n_\mu=1}^{N_\mu} e^{\frac{2\pi i}{N_\mu}\left(k_\mu - k_\mu^{'}\right)n_\mu} = \delta_{k_\mu + k_\mu^{'}, 0} + \eta_\mu \delta_{k_\mu + k_\mu^{'}, N_\mu}. \tag{271}$$

Then the Fourier transform for the lattice field $\Phi_{J(l,i)}$ and its inverse transform can be written as

$$\Phi_{J(l,i)} = \frac{1}{\sqrt{N_\sigma N_\beta}} \sum_{k_x=k_{x1}}^{k_{x2}} \sum_{k_\beta=k_{\beta 1}}^{k_{\beta 2}} f_{I(k_x,k_\beta)} e^{i(a_\sigma p_x l + a_\beta p_\beta i)}, \tag{272}$$

$$f_{I(k_x,k_\beta)} = \frac{1}{\sqrt{N_\sigma N_\beta}} \sum_{l=1}^{N_\sigma} \sum_{i=1}^{N_\beta} \Phi_{J(l,i)} e^{-i(a_\sigma p_x l + a_\beta p_\beta i)}, \tag{273}$$

where the index $I(k_x, k_\beta) \equiv N_\beta(k_x - k_{x1}) + k_\beta - k_{\beta 1} + 1$ is the number of the site on the lattice in the momentum space and $p_x = 2\pi k_x / a_\sigma N_\sigma$ and $p_\beta = 2\pi k_\beta / a_\beta N_\beta$. Let's check if this transformations are correct, for that, we merely substitute eq. (273) into eq. (272). If the transformations are correct, we will obtain $\Phi_{J(l,i)}$

$$\begin{aligned} \Phi_{J(l,i)} &= \frac{1}{N_\sigma N_\beta} \sum_{l'=1}^{N_\sigma} \sum_{i'=1}^{N_\beta} \Phi_{J(l',i')} \sum_{k_x=k_{x1}}^{k_{x2}} \sum_{k_\beta=k_{\beta 1}}^{k_{\beta 2}} e^{-i\left(a_\sigma p_x(l'-l) + a_\beta p_\beta(i'-i)\right)} \\ &= \sum_{l'=1}^{N_\sigma} \sum_{i'=1}^{N_\beta} \Phi_{J(l',i')} \delta_{l',l} \delta_{i',i} = \Phi_{J(l,i)}. \end{aligned} \tag{274}$$

The neutral scalar field is real, so $\Phi_J^* = \Phi_J$. Therefore, the complex function $f_{I(k_x,k_\beta)}$ can be represented by its amplitude and phase in the form

$$f_{I(k_x,k_\beta)} = R_{I(k_x,k_\beta)} e^{i(a_\sigma p_x + a_\beta p_\beta)}. \tag{275}$$

Substituting this into eq. (272), we have

$$\Phi_{J(l,i)} = \frac{1}{\sqrt{N_\sigma N_\beta}} \sum_{k_x=k_{x1}}^{k_{x2}} \sum_{k_\beta=k_{\beta 1}}^{k_{\beta 2}} R_{I(k_x,k_\beta)} e^{i[a_\sigma p_x(1+l) + a_\beta p_\beta(1+i)]} \tag{276}$$

and into eq. (273)

$$R_{I(k_x,k_\beta)} = \frac{1}{\sqrt{N_\sigma N_\beta}} \sum_{l=1}^{N_\sigma} \sum_{i=1}^{N_\beta} \Phi_{J(l,i)} e^{-i[a_\sigma p_x(1+l) + a_\beta p_\beta(1+i)]}. \tag{277}$$

And so we can rewrite eq. (254) as

$$\mathcal{Z}_{lat} = \pi^{-\frac{N_\sigma N_\beta}{2}} |det\mathcal{J}| \prod_{I=1}^{N_\sigma N_\beta} dR_I exp\left\{ - \sum_{k_x=k_{x1}}^{k_{x2}} \sum_{k_\beta=k_{\beta 1}}^{k_{\beta 2}} \sum_{k'_x=k_{x1}}^{k_{x2}} \sum_{k'_\beta=k_{\beta 1}}^{k_{\beta 2}} R_{I(k_x,k_\beta)} R_{I(k'_x,k'_\beta)} B_{I(k_x,k_\beta),I(k'_x,k'_\beta)} \right\}, \tag{278}$$

where $\mathcal{J}$ is the Jacobian matrix and

$$B_{I(k_x,k_\beta),I(k'_x,k'_\beta)} = \sum_{J=1}^{N_\sigma N_\beta} \sum_{J'=1}^{N_\sigma N_\beta} \frac{A_{J,J'}}{N_\sigma N_\beta} exp\left\{ i \left[a_\sigma [p_x(1+l) + p'_x(1+l')] + a_\beta [p_\beta(1+i) + p'_\beta(1+i')] \right] \right\}, \tag{279}$$

$$\begin{aligned} A_{J,J'} &= \left(2\left(1 + \frac{a_\beta^2}{a_\sigma^2}\right) + a_\beta^2 m^2 \right) \delta_{l',l} \delta_{i,i'} \\ &- \delta_{l,l'} \left[(1 - \delta_{i,N_\beta}) \delta_{i',i+1} + \delta_{i,N_\beta} \delta_{i',1} + (1 - \delta_{i',N_\beta}) \delta_{i,i'+1} + \delta_{i,N_\beta} \delta_{i,1} \right] \\ &- \frac{a_\beta^2}{a_\sigma^2} \delta_{i,i'} \left[(1 - \delta_{l,N_\sigma}) \delta_{l',l+1} + \delta_{l,N_\sigma} \delta_{l',1} + (1 - \delta_{l',N_\sigma}) \delta_{l,l'+1} + \delta_{l',N_\sigma} \delta_{l,1} \right]. \end{aligned} \tag{280}$$

We can note that $B_{I(k_x,k_\beta),I(k'_x,k'_\beta)}$ can be separated into three terms, let's calculate every term separately. Starting with the first term

$$\frac{1}{N_\sigma N_\beta}\sum_{J,J'=1}^{N_\sigma N_\beta}\left(2\left(1+\frac{a_\beta^2}{a_\sigma^2}\right)+a_\beta^2m^2\right)\delta_{l',l}\delta_{i,i'}e^{i\left[a_\sigma[p_x(1+l)+p'_x(1+l')]+a_\beta[p_\beta(1+i)p'_\beta(1+i')]\right]}$$
$$=\frac{1}{N_\sigma N_\beta}\sum_{J=1}^{N_\sigma N_\beta}\left(2\left(1+\frac{a_\beta^2}{a_\sigma^2}\right)+a_\beta^2m^2\right)e^{i[a_\sigma(1+l)(p_x+p'_x)+a_\beta(1+i)(p_\beta+p'_\beta)]}$$
$$=\frac{1}{N_\sigma N_\beta}\left(2\left(1+\frac{a_\beta^2}{a_\sigma^2}\right)+a_\beta^2m^2\right)\sum_{l=1}^{N_\sigma}e^{i[a_\sigma(1+l)(p_x+p'_x)]}\sum_{i=1}^{N_\beta}e^{i[a_\beta(1+i)(p_\beta+p'_\beta)]}$$
$$=\left(2\left(1+\frac{a_\beta^2}{a_\sigma^2}\right)+a_\beta^2m^2\right)\left(\delta_{k_x+k'_x,0}+\eta_x\delta_{k_x+k'_x,N_\sigma}\right)\left(\delta_{k_\beta+k'_\beta,0}+\eta_\beta\delta_{k_\beta+k'_\beta,N_\beta}\right), \tag{281}$$

where we used the relation given by eq. (271).

Now we take the second term of $B_{I(k_x,k_\beta),I(k'_x,k'_\beta)}$:

$$\frac{1}{N_\sigma N_\beta}\sum_{J,J'=1}^{N_\sigma N_\beta}\delta_{l,l'}\left[(1-\delta_{i,N_\beta})\delta_{i',i+1}+\delta_{i,N_\beta}\delta_{i',1}+(1-\delta_{i',N_\beta})\delta_{i,i'+1}+\delta_{i',N_\beta}\delta_{i,1}\right]e^{i\left[a_\sigma[p_x(1+l)+p'_x(1+l')]+a_\beta[p_\beta(1+i)p'_\beta(1+i')]\right]}$$
$$=\frac{1}{N_\sigma N_\beta}\sum_{l}^{N_\sigma}\sum_{i,i'=1}^{N_\beta}\left[(1-\delta_{i,N_\beta})\delta_{i',i+1}+\delta_{i,N_\beta}\delta_{i',1}+(1-\delta_{i',N_\beta})\delta_{i,i'+1}+\delta_{i',N_\beta}\delta_{i,1}\right]e^{i\left[a_\sigma(1+l)([p_x+p'_x)]+a_\beta[p_\beta(1+i)p'_\beta(1+i')]\right]}$$
$$=\left(\delta_{k_x+k'_x,0}+\eta_x\delta_{k_x+k'_x,N_\sigma}\right)\Bigg\{\left(\delta_{k_\beta+k'_\beta,0}+\eta_\beta\delta_{k_\beta+k'_\beta,N_\beta}\right)e^{i\left[a_\sigma(p_x+p'_x)+a_\beta(p_\beta+2p'_\beta)\right]}$$
$$-e^{i\left[a_\sigma(p_x+p'_x)+a_\beta[p_\beta(1+N_\beta)+p'_\beta(2+N_\beta)]\right]}+e^{i\left[a_\sigma(p_x+p'_x)+a_\beta[p_\beta(1+N_\beta)+2p'_\beta]\right]}+\left(\delta_{k_\beta+k'_\beta,0}+\eta_\beta\delta_{k_\beta+k'_\beta,N_\beta}\right)e^{i[a_\sigma(p_x+p'_x)+a_\beta p_\beta]}$$
$$-e^{i\left[a_\sigma(p_x+p'_x)+a_\beta[p_\beta(2+N_\beta)+p'_\beta(1+N_\beta)]\right]}+e^{i\left[a_\sigma(p_x+p'_x)+a_\beta[2p_\beta+p'_\beta(1+N_\beta)]\right]}\Bigg\}$$
$$=\left(\delta_{k_x+k'_x,0}+\eta_x\delta_{k_x+k'_x,N_\sigma}\right)\left(\delta_{k_\beta+k'_\beta,0}+\eta_\beta\delta_{k_\beta+k'_\beta,N_\beta}\right)\left(e^{ia_\beta p'_\beta}+e^{ia_\beta p_\beta}\right)$$
$$+\left(\delta_{k_\beta+k'_\beta,0}+\eta_\beta\delta_{k_\beta+k'_\beta,N_\beta}\right)\left(-e^{ia_\beta(p_\beta(1+N_\beta)+p'_\beta(2+N_\beta)}+e^{ia_\beta(p_\beta(1+N_\beta)+2p'_\beta)}-e^{ia_\beta(p_\beta(2+N_\beta)+p'_\beta(1+N_\beta)}+e^{ia_\beta(2p_\beta+p'_\beta(1+N_\beta)}\right)$$
$$=\left(\delta_{k_x+k'_x,0}+\eta_x\delta_{k_x+k'_x,N_\sigma}\right)\left(\delta_{k_\beta+k'_\beta,0}+\eta_\beta\delta_{k_\beta+k'_\beta,N_\beta}\right)\left(e^{-ia_\beta p_\beta}+e^{ia_\beta p_\beta}\right)$$
$$=2\left(\delta_{k_x+k'_x,0}+\eta_x\delta_{k_x+k'_x,N_\sigma}\right)\left(\delta_{k_\beta+k'_\beta,0}+\eta_\beta\delta_{k_\beta+k'_\beta,N_\beta}\right)cos(a_\beta p_\beta). \tag{282}$$

For the third term of $B_{I(k_x,k_\beta),I(k'_x,k'_\beta)}$ the calculations are very similar:

$$\frac{1}{N_\sigma N_\beta}\frac{a_\beta^2}{a_\sigma^2}\sum_{J,J'=1}^{N_\sigma N_\beta}\delta_{i,i'}\left[(1-\delta_{l,N_\sigma})\delta_{l',l+1}+\delta_{l,N_\sigma}\delta_{l',1}+(1-\delta_{l',N_\sigma})\delta_{l,l'+1}+\delta_{l',N_\sigma}\delta_{l,1}\right]e^{i\left[a_\sigma[p_x(1+l)+p'_x(1+l')]+a_\beta[p_\beta(1+i)p'_\beta(1+i')]\right]}$$
$$=\frac{2a_\beta^2}{a_\sigma^2}\left(\delta_{k_x+k'_x,0}+\eta_x\delta_{k_x+k'_x,N_\sigma}\right)\left(\delta_{k_\beta+k'_\beta,0}+\eta_\beta\delta_{k_\beta+k'_\beta,N_\beta}\right)cos(a_\sigma p_x). \tag{283}$$

Substituting eqs. (281), (282) and (283) into (279) and (280), we have

$$B_{I(k_x,k_\beta),I(k'_x,k'_\beta)} = \left[\left(2\left(1+\frac{a_\beta^2}{a_\sigma^2}\right)+a_\beta^2 m^2\right) - 2cos(a_\beta p_\beta) - \frac{2a_\beta^2}{a_\sigma^2}cos(a_\sigma p_x)\right]$$
$$\left(\delta_{k_x+k'_x,0} + \eta_x \delta_{k_x+k'_x,N_\sigma}\right)\left(\delta_{k_\beta+k'_\beta,0} + \eta_\beta \delta_{k_\beta+k'_\beta,N_\beta}\right)$$
$$= a_\beta^2\left[\left(\frac{2}{a_\beta}sin\left(\frac{a_\beta p_\beta}{2}\right)\right)^2 + \left(\frac{2}{a_\sigma}sin\left(\frac{a_\sigma p_x}{2}\right)\right)^2 + m^2\right]$$
$$\left(\delta_{k_x+k'_x,0} + \eta_x \delta_{k_x+k'_x,N_\sigma}\right)\left(\delta_{k_\beta+k'_\beta,0} + \eta_\beta \delta_{k_\beta+k'_\beta,N_\beta}\right), \tag{284}$$

where we used the trigonometric identity $cos(2\theta) = 1 - 2sin^2(\theta)$. We can rewrite this equation as

$$B_{I(k_x,k_\beta),I(k'_x,k'_\beta)} = G_{I(k_x,k_\beta)}\left(\delta_{k_x+k'_x,0} + \eta_x \delta_{k_x+k'_x,N_\sigma}\right)\left(\delta_{k_\beta+k'_\beta,0} + \eta_\beta \delta_{k_\beta+k'_\beta,N_\beta}\right), \tag{285}$$

where

$$G_{I(k_x,k_\beta)} = a_\beta^2\left[\left(\frac{2}{a_\beta}sin\left(\frac{a_\beta p_\beta}{2}\right)\right)^2 + \left(\frac{2}{a_\sigma}sin\left(\frac{a_\sigma p_x}{2}\right)\right)^2 + m^2\right]. \tag{286}$$

To solve integral of eq. (278) we can use again the formula for Riemann integrals given by eq. (255) and we have

$$\mathcal{Z}_{lat} = \frac{|det\mathcal{J}|}{\sqrt{detB}}. \tag{287}$$

The matrix B has $N_\sigma N_\beta$ nonzero elements equal to $G_{I(k_x,k_\beta)}$. In any row and any column there is only one nonzero element. Therefore, the determinant of the matrix B can be written as

$$detB = \chi \prod_{k_x=k_{x1}}^{k_{x2}} \prod_{k_\beta=k_{\beta 1}}^{k_{\beta 2}} G_{I(k_x,k_\beta)}, \tag{288}$$

where χ is a factor that can be calculated but we did not calculate it here.

If we substitute eq. (288) into eq. (287) and use the fact that $|det\mathcal{J}|/\sqrt{\chi} = 1$, we have

$$\mathcal{Z}_{lat} = \frac{1}{\sqrt{\prod_{k_x=k_{x1}}^{k_{x2}} \prod_{k_\beta=k_{\beta 1}}^{k_{\beta 2}} G_{I(k_x,k_\beta)}}} \tag{289}$$

which is the result for the $\mathcal{Z}_{lat}$ in one spatial dimension of the momentum space. If we generalize this result to three spatial dimensions, we have

$$\mathcal{Z}_{lat} = \frac{1}{\sqrt{\prod_{\boldsymbol{k}=k_{\sigma 1}}^{k_{\sigma 2}} \prod_{k_\beta=k_{\beta 1}}^{k_{\beta 2}} G_{I(\boldsymbol{k},k_\beta)}}}, \tag{290}$$

$$G_{I(\boldsymbol{k},k_\beta)} = a_\beta^2\left[\left(\frac{2}{a_\beta}sin\left(\frac{a_\beta p_\beta}{2}\right)\right)^2 + \sum_{\alpha=1}^{3}\left(\frac{2}{a_\sigma}sin\left(\frac{a_\sigma p_\alpha}{2}\right)\right)^2 + m^2.\right] \tag{291}$$

C. Thermodynamic Quantities on the Lattice

The thermodynamic quantities on the lattice can now be derived from the partition function (290). The thermodynamic potential is

$$\begin{aligned}\Omega_{lat} &= -Tln\mathcal{Z}_{lat} = \frac{T}{2}\sum_{\boldsymbol{k}=k_{\sigma1}}^{k_{\sigma2}}\sum_{k_\beta=k_{\beta1}}^{k_{\beta2}} ln\left[G_{I(\boldsymbol{k},k_\beta)}\right] \\ &= \frac{T}{2}\sum_{\boldsymbol{k}=k_{\sigma1}}^{k_{\sigma2}}\sum_{k_\beta=k_{\beta1}}^{k_{\beta2}} ln\Bigg\{a_\beta^2\Bigg[\left(\frac{2}{a_\beta}sin\left(\frac{a_\beta p_\beta}{2}\right)\right)^2+\sum_{\alpha=1}^{3}\left(\frac{2}{a_\sigma}sin\left(\frac{a_\sigma p_\alpha}{2}\right)\right)^2+m^2\Bigg]\Bigg\}. \end{aligned} \tag{292}$$

The energy on the lattice is

$$\begin{aligned}E_{lat} &= -\frac{\partial}{\partial\beta}ln\mathcal{Z}_{lat} = -\frac{1}{N_\beta}\frac{\partial}{\partial a_\beta}ln\mathcal{Z}_{lat} \\ &= \frac{1}{2N_\beta}\frac{\partial}{\partial a_\beta}\sum_{\boldsymbol{k}=k_{\sigma1}}^{k_{\sigma2}}\sum_{k_\beta=k_{\beta1}}^{k_{\beta2}} ln\Bigg\{a_\beta^2\Bigg[\left(\frac{2}{a_\beta}sin\left(\frac{a_\beta p_\beta}{2}\right)\right)^2+\sum_{\alpha=1}^{3}\left(\frac{2}{a_\sigma}sin\left(\frac{a_\sigma p_\alpha}{2}\right)\right)^2+m^2\Bigg]\Bigg\} \\ &= \frac{1}{\beta}\sum_{\boldsymbol{k}=k_{\sigma1}}^{k_{\sigma2}}\sum_{k_\beta=k_{\beta1}}^{k_{\beta2}} \frac{\sum_{\alpha=1}^{3}\left(\frac{2}{a_\sigma}sin\left(\frac{a_\sigma p_\alpha}{2}\right)\right)^2+m^2}{\left(\frac{2}{a_\beta}sin\left(\frac{a_\beta p_\beta}{2}\right)\right)^2+\sum_{\alpha=1}^{3}\left(\frac{2}{a_\sigma}sin\left(\frac{a_\sigma p_\alpha}{2}\right)\right)^2+m^2}. \end{aligned} \tag{293}$$

To calculate the pressure we have to remember that $V = L_\sigma^3$ and $L_\sigma = a_\sigma N_\sigma$. The pressure is given by

$$\begin{aligned}P_{lat} &= \frac{1}{\beta}\frac{\partial}{\partial V}ln\mathcal{Z}_{lat} \\ &= -\frac{1}{2\beta}\frac{V^{-2/3}}{3N_\sigma}\frac{\partial}{\partial a_\sigma}\sum_{\boldsymbol{k}=k_{\sigma1}}^{k_{\sigma2}}\sum_{k_\beta=k_{\beta1}}^{k_{\beta2}} ln\Bigg\{a_\beta^2\Bigg[\left(\frac{2}{a_\beta}sin\left(\frac{a_\beta p_\beta}{2}\right)\right)^2+\sum_{\alpha=1}^{3}\left(\frac{2}{a_\sigma}sin\left(\frac{a_\sigma p_\alpha}{2}\right)\right)^2+m^2\Bigg]\Bigg\} \\ &= \frac{1}{3\beta V}\sum_{\boldsymbol{k}=k_{\sigma1}}^{k_{\sigma2}}\sum_{k_\beta=k_{\beta1}}^{k_{\beta2}} \frac{\sum_{\alpha=1}^{3}\left(\frac{2}{a_\sigma}sin\left(\frac{a_\sigma p_\alpha}{2}\right)\right)^2}{\left(\frac{2}{a_\beta}sin\left(\frac{a_\beta p_\beta}{2}\right)\right)^2+\sum_{\alpha=1}^{3}\left(\frac{2}{a_\sigma}sin\left(\frac{a_\sigma p_\alpha}{2}\right)\right)^2+m^2}. \end{aligned} \tag{294}$$

D. Continuum Limit

In this section we want to find the Energy and the Pressure in the continuum limit, which means that $N_\sigma \to \infty$, $a_\sigma \to 0$ at $L_\sigma = const$ and $N_\beta \to \infty$, $a_\beta \to 0$ at $\beta = const$.

In order to calculate this limits, it is more interesting to write eqs. (293) and (294) in terms of k_μ:

$$E_{lat} = \frac{1}{\beta}\sum_{\boldsymbol{k}=k_{\sigma1}}^{k_{\sigma2}}\sum_{k_\beta=k_{\beta1}}^{k_{\beta2}} \frac{\sum_{\alpha=1}^{3}\left(\frac{2}{a_\sigma}sin\left(\frac{\pi k_\alpha}{N_\sigma}\right)\right)^2+m^2}{\left(\frac{2}{a_\beta}sin\left(\frac{\pi k_\beta}{N_\beta}\right)\right)^2+\sum_{\alpha=1}^{3}\left(\frac{2}{a_\sigma}sin\left(\frac{\pi k_\alpha}{N_\sigma}\right)\right)^2+m^2}, \tag{295}$$

$$P_{lat} = \frac{1}{3\beta V}\sum_{\boldsymbol{k}=k_{\sigma1}}^{k_{\sigma2}}\sum_{k_\beta=k_{\beta1}}^{k_{\beta2}} \frac{\sum_{\alpha=1}^{3}\left(\frac{2}{a_\sigma}sin\left(\frac{\pi k_\alpha}{N_\sigma}\right)\right)^2}{\left(\frac{2}{a_\beta}sin\left(\frac{\pi k_\beta}{N_\beta}\right)\right)^2+\sum_{\alpha=1}^{3}\left(\frac{2}{a_\sigma}sin\left(\frac{\pi k_\alpha}{N_\sigma}\right)\right)^2+m^2}. \tag{296}$$

As $N_\sigma \to \infty$ and $N_\beta \to \infty$ we have that the argument of the sine function is very small and so we can use the approximation $sin(x) = x$. Also, from eqs. (263) and (264) it is easy to see that in the continuum limit $k_{\sigma 1} = k_{\beta 1} \to -\infty$ and $k_{\sigma 2} = k_{\beta 2} \to +\infty$.

For the energy we have

$$E = \lim_{N_\sigma, N_\beta \to \infty} E_{lat} = \frac{1}{\beta} \sum_{\boldsymbol{k}=-\infty}^{\infty} \sum_{k_\beta=-\infty}^{\infty} \frac{\sum_{\alpha=1}^{3} \left(\frac{2\pi k_\alpha}{L_\sigma}\right)^2 + m^2}{\left(\frac{2\pi k_\beta}{\beta}\right)^2 + \sum_{\alpha=1}^{3} \left(\frac{2\pi k_\alpha}{L_\sigma}\right)^2 + m^2} \tag{297}$$

Using the generating function

$$\sum_{k=-\infty}^{\infty} \frac{a^2}{(2\pi k)^2 + a^2} = \frac{a}{2} coth \frac{a}{2} \tag{298}$$

we obtain

$$E = \frac{1}{\beta} \sum_{\boldsymbol{k}=-\infty}^{\infty} \sum_{k_\beta=-\infty}^{\infty} \frac{\beta^2 D^2}{(2\pi k_\beta)^2 + \beta^2 D^2} = \sum_{\boldsymbol{k}=-\infty}^{\infty} \frac{D}{2} coth \frac{\beta D}{2}, \tag{299}$$

where

$$D^2 = \sum_{\alpha=1}^{3} \left(\frac{2\pi k_\alpha}{L_\sigma}\right)^2 + m^2. \tag{300}$$

Remembering that $coth(x) = (e^x + e^{-x})/2$, we can continue our calculations

$$\begin{aligned} E &= \sum_{\boldsymbol{k}=-\infty}^{\infty} \frac{D}{2} \frac{e^{\beta D/2} + e^{-\beta D/2}}{e^{\beta D/2} - e^{-\beta D/2}} = \sum_{\boldsymbol{k}=-\infty}^{\infty} \frac{D}{2} \frac{e^{\beta D} + 1}{e^{\beta D} - 1} \\ &= \sum_{\boldsymbol{k}=-\infty}^{\infty} \frac{D}{2} \left(1 + \frac{2}{e^{\beta D} - 1}\right) = \sum_{\boldsymbol{k}=-\infty}^{\infty} \frac{D}{2} + \sum_{\boldsymbol{k}=-\infty}^{\infty} \frac{D}{e^{\beta D} - 1}. \end{aligned} \tag{301}$$

Let us take a closer look at D

$$D = \left[\sum_{\alpha=1}^{3} \left(\frac{2\pi k_\alpha}{L_\sigma}\right)^2 + m^2\right]^{1/2} = \left[\sum_{\alpha=1}^{3} p_\alpha^2 + m^2\right]^{1/2} = \sqrt{\boldsymbol{p}^2 + m^2} = E_{\boldsymbol{p}}. \tag{302}$$

Substituting this into eq. (301), we obtain

$$E = \sum_{\boldsymbol{k}=-\infty}^{\infty} \frac{E_{\boldsymbol{p}}}{2} + \sum_{\boldsymbol{k}=-\infty}^{\infty} \frac{E_{\boldsymbol{p}}}{e^{\beta E_{\boldsymbol{p}}} - 1} \tag{303}$$

which is the same result as obtained through the method of the second quantization (see eq.(212)).

For the pressure we have very similar calculations. If we observe the similarities between eqs. (295) and (296), it is easy to see that

$$P = \frac{1}{3\beta V} \sum_{\boldsymbol{k}=-\infty}^{\infty} \sum_{k_\beta=-\infty}^{\infty} \frac{D^2 - m^2}{\left(\frac{2\pi k_\beta}{\beta}\right)^2 + D^2} = \frac{1}{3\beta V} \sum_{\boldsymbol{k}=-\infty}^{\infty} \sum_{k_\beta=-\infty}^{\infty} \frac{\beta^2 D^2 - \beta^2 m^2}{(2\pi k_\beta)^2 + \beta^2 D^2}. \tag{304}$$

Now, if we modify eq.(298) a little

$$\sum_{k=-\infty}^{\infty}\frac{1}{(2\pi k)^2+a^2}=\frac{1}{2a}coth\frac{a}{2} \tag{305}$$

we obtain

$$\begin{aligned} P &= \frac{1}{3\beta V}\sum_{\boldsymbol{k}=-\infty}^{\infty}(\beta^2 D^2-\beta^2 m^2)\frac{1}{2\beta D}coth\left(\frac{\beta D}{2}\right) \\ &= \frac{1}{3\beta V}\sum_{\boldsymbol{k}=-\infty}^{\infty}\frac{\beta^2 D^2-\beta^2 m^2}{2\beta D}\frac{e^{\beta D}+1}{e^{\beta D}-1} \\ &= \frac{1}{6V}\sum_{\boldsymbol{k}=-\infty}^{\infty}\frac{E_{\boldsymbol{p}}^2-m^2}{E_{\boldsymbol{p}}}+\frac{1}{3V}\sum_{\boldsymbol{k}=-\infty}^{\infty}\frac{E_{\boldsymbol{p}}^2-m^2}{E_{\boldsymbol{p}}}\frac{1}{e^{\beta E_{\boldsymbol{p}}}-1}. \end{aligned} \tag{306}$$

If we observe that $(E_{\boldsymbol{p}}^2-m^2)/E_{\boldsymbol{p}}=\boldsymbol{p}^2/E_{\boldsymbol{p}}$, we obtain

$$P=\frac{1}{3V}\sum_{\boldsymbol{p}}\frac{\boldsymbol{p}^2}{E_{\boldsymbol{p}}}+\frac{1}{3V}\sum_{\boldsymbol{p}}\frac{\boldsymbol{p}^2}{E_{\boldsymbol{p}}}\frac{1}{(e^{\beta E_{\boldsymbol{p}}}-1)} \tag{307}$$

which is equal to the result obtained through the second quantization method (see eq. (220)).

VI. CONCLUSIONS

The partition function for the real scalar field in a finite volume was obtained by the canonical quantization method and the result coincides exactly with Eq. (8.8) from Ref. [4]. Also, the main thermodynamic quantities obtained are in accordance with the ones obtained in the same reference.

The partition function obtained by the path integral method is in accordance with the one obtained in Ref. [6]. We have also found the exact analytical expressions for some thermodynamic quantities, which are also in accordance to the ones obtained in Ref. [6].

The comparison between the thermodynamic quantities obtained by the canonical quantization method and by the path integral method was made and the results agree completely. In both cases the thermodynamic quantities can be separated into two quantities: a sum over the vacuum and a physical term. The vacuum term is divergent. Fortunately, from a practical point of view this infinity is rather harmless. Since physical observable involve differences and not the absolute value of the quantity, the vacuum term always drops out.

Acknowledgments

I would like to convey my gratefulness to my supervisor in this Summer Student Programme, Alexandru S. Parvan, first of all for accepting me as a student. And also for his patience, his support and the excellent guidance. The knowledge that I have gained throughout this six weeks I perceive as highly valuable component in my future career development.

I would especially like to thank my supervisor in Brazil, Débora Peres Menezes, for encouraging me and supporting me to take part in the Programme.

My thanks and appreciation also to the JINR University Centre for providing me such a nice opportunity to work in one of the JINR's laboratories and for financial support.

I also thank my family for always supporting me in all possible ways.

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