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BOGOLIUBOV LABORATORY OF THEORETICAL PHYSICS

Final report on the summer student programme

Isorotating $\mathbb{C}P^2$ *solitons*

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Abstract

In this report the results of studying uniformly isorotating radially symmetric $\mathbb{C}P^2$ solutions in (2+1) dimensions with classical Lagrangian are presented and discussed. The region of permitted frequences where the solitons are stable is found both analytically and numerically. Special attention is paid to relations between Noether charges and angular momenta of the solutions. The several cases of fields with different n_1, n_2 quantum numbers were analyzed from the point of view of their dynamical characteristics.

1 Theoretical study

Generally speaking the $\mathbb{C}P^n$ symmetry [1] implies that the Lagrangian of the model possesses the Lie group of symmetry isomorphic to n-dimensional complex projective plane, which is a homogeneous space. The group can be obtained by regarding this space as a factor-space

$$\mathbb{CP}^n = \mathbb{C}^{n+1}/\mathbb{C}* \tag{1}$$

where operation is a product $*$ of complex vector $\mathbb{C}^{n+1} \ni \vec{z} = (z_1, \ldots, z_{n+1})$, $z_j \in \mathbb{C},\ j = \overline{1, n-1}$ and complex number, defining equivalence classes $\vec{z} \sim \xi \cdot \vec{z}$, where $\xi \in \mathbb{C} \setminus \{0\}$. Considering an odd-dimensional sphere equation $\vec{z}^\dagger \vec{z} = 1$ and decomposing ξ into trigonometric form of complex number, one can find that if n=2

$$\mathbb{CP}^2 = \frac{S^5}{U(1)} = \frac{SU(3)}{SU(2) \times U(1)} \tag{2}$$

There is a number of peer ways to formulate $\mathbb{C}P^2$ model [1], [4]. The first one that we need is description via 8-component color field $\mathfrak{n}$, which [5] either can be decomposed in Gell-Mann matrices basis with real parameters $\mathfrak{n} = n^a \lambda_a$, $a = \overline{1, 8}$; or regarded as a matrix product $\mathfrak{n} = U^\dagger \lambda_8 U$, where $U \in SU(3)$. Then the Lagrangian takes form

$$\mathcal{L} = \frac{1}{4} \cdot Tr\left(\partial_\mu \mathfrak{n}\, \partial^\mu \mathfrak{n}\right) - V \tag{3}$$

Hereinafter greek indices $\mu = \overline{0, 2}$ and metric tensor is taken as standard Minkowski metric, i.e. $g_{\mu\nu} = \eta_{\mu\nu} = diag(-1, +1, +1)$. The (3) is invariant under global transformations $\mathfrak{n} \to W^\dagger \mathfrak{n} W$, where $W \in U(3)$, and also has an invariant subgroup $U \to H \cdot U$, where $SU(2) \times U(1) \supset H = \{h : [h, \lambda_8] = 0\}$.

The second way of obtaining the Lagrangian is by using the homogeneous coordinates Z on $\mathbb{C}P^2$, which will be applied further. The components of the color field $\mathfrak{n}$ can be introduced as $n^a = Z^\dagger \lambda^a Z$, thereby the Lagrangian (3) takes a new form

$$\mathcal{L} = 2 \cdot (D_\mu Z)^\dagger D^\mu Z - V \tag{4}$$

where $D_\mu Z = \partial_\mu Z - \left(Z^\dagger \partial_\mu Z\right) \cdot Z$ and Z is a complex unimodular vector with arbitrary phase. The potential [4] can be chosen as

$$V = \mu^2 \cdot \left(\left(n^3\right)^2 + \left(n^8\right)^2 - \frac{4}{3}\right) \tag{5}$$

As soon as we are considering monosolitonic solutions with axial symmetry, we can impose an ansatz [5] of the form

$$Z = \left(cos\, F\,(r)\,,\; sin\, F\,(r) \cdot cos\, G\,(r) \cdot e^{i\varphi},\; sin\, F\,(r) \cdot sin\, G\,(r) \cdot e^{i\psi}\right) \tag{6}$$

where $\varphi = \omega_1 t + n_1 \vartheta$, $\psi = \omega_2 t + n_2 \vartheta$, $\{r, \vartheta\}$ are polar coordinates on physical space $\mathbb{R}^2$; ω_1, ω_2 are real numbers and n_1, n_2 are positive integers. The boundary conditions imposed on profile functions are:

$$\begin{aligned} F\,(0) &= 0, & F\,(\infty) &= \frac{\pi}{2} \\ G\,(0) &= \frac{\pi}{2}, & G\,(\infty) &= 0 \end{aligned} \tag{7}$$

In terms of (6) the potential takes form

$$V = \frac{1}{4}\mu^2 \sin^2 F \cdot \left(2\sin^2 F \cos 4G - 7\cos 2F - 9\right) \tag{8}$$

Where μ^2 is mass parameter, which was set to 1 in our research. According to (7) potential vanishes at spatial infinity $\lim\limits_{r\to\infty} V = 0$.

Energy density component of energy-momentum tensor [2]

$$T_{00} = (D_0 Z)^\dagger (D_0 Z) + (D_j Z)^\dagger (D_j Z) + V \tag{9}$$

with regard to pofile functions parametrization can be written as

$$
\begin{aligned}
T_{00} = E = {} & F'^2 + \sin^2 F G'^2 \\
& + \frac{1}{16r^2}\sin^2 F \cdot [2\left(\left(n_1^2 - n_2^2\right) + r^2\left(\omega_1^2 - \omega_2^2\right)\right) \cdot \left(4cos^2F\, cos2G - sin^2F\, cos4G\right) \\
& + \left(\left(n_1^2 + n_2^2\right) + r^2\left(\omega_1^2 + \omega_2^2\right)\right) \cdot (3cos2F + 5) \\
& - 2sin^2F \cdot \left(n_1 n_2 + r^2\omega_1\omega_2\right)]
\end{aligned}
\tag{10}
$$

Angular momentum density of solitonic configuration can also be obtained from E-M tensor as a time-angular component in polar coordinates [6]

$$
T_{0\varphi} = \frac{\partial \mathcal{L}}{\partial\left(\partial_0 Z\right)}\partial_\varphi Z + \partial_\varphi Z^\dagger \frac{\partial \mathcal{L}}{\partial\left(\partial_0 Z^\dagger\right)} \tag{11}
$$

Concerning the (6), one can get

$$
\begin{aligned}
T_{0\varphi} = {} & 2\,\omega_1 \sin^2 F\left(n_1\cos^2 G\left(1 - \sin^2 F\cos^2 G\right) - n_2\sin^2 F\sin^2 G\cos^2 G\right) \\
& + 2\,\omega_2 \sin^2 F\left(n_2\sin^2 G\left(1 - \sin^2 F\sin^2 G\right) - n_1\sin^2 F\sin^2 G\cos^2 G\right)
\end{aligned}
\tag{12}
$$

This expression can be separated into two components by grouping multipliers near angular frequencies. This multipliers can be regarded as corresponding inertia momenta density of solution.

Due to the existance of symmetry (2), the model possesses a conserving Noether current [2]

$$
j_\mu = \frac{i}{2}\left(\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu Z\right)}\Lambda Z - Z^\dagger \Lambda \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu Z^\dagger\right)}\right) \tag{13}
$$

Where Λ is linear combination of Cartan subalgebra generators. In this paper it is taken as $\Lambda = \left(\lambda_3 + \sqrt{3}\lambda_8\right)/2$. Imposing the radial symmetry ansatz, one can find the time component

$$
\begin{aligned}
j_0 = Q = {} & \frac{1}{16}\omega_1 \sin^2 F\left(-8\cos^2 F\cos 2G - 2\sin^2 F\cos 4G - 5\cos 2F - 3\right) \\
& + \frac{1}{16}\omega_2 \sin^2 F\left(16\cos^2 F\cos 2G + 2\sin^2 F\cos 4G - 7\cos 2F - 9\right)
\end{aligned}
\tag{14}
$$

Here we can observe the same situation as with angular momentum, namely the charge can be divided into two terms with multipliers of respective angular frequencies.

The field equations of the model can be obtained by standard procedure of action, corresponding to (4) minimization. Then the Euler-Lagrange equations are

$$\partial_r \frac{\partial}{\partial F_r'}\left(\sqrt{-g}\mathcal{L}\right) - \frac{\partial\left(\sqrt{-g}\mathcal{L}\right)}{\partial F} = 0; \quad \partial_r \frac{\partial}{\partial G_r'}\left(\sqrt{-g}\mathcal{L}\right) - \frac{\partial\left(\sqrt{-g}\mathcal{L}\right)}{\partial G} = 0 \quad (15)$$

Where $g = \det(g_{\mu\nu}(r))$ and prime stands for derivative. After the simplification procedure the asymptotical behaviour of the solutions can be studied. To that end it is necessary to introduce a change of variables

$$F(r) \to \frac{\pi}{2} - \mathfrak{f}(r); \quad G(r) \to \mathfrak{g}(r) \quad (16)$$

Where the new fields satisfy the condition of vanishing at spatial infinity, i.e.

$$\lim_{r\to\infty} \mathfrak{f}(r) = 0; \quad \lim_{r\to\infty} \mathfrak{g}(r) = 0 \quad (17)$$

Then full linearization of the equations (15) should be carried out, it means that all the functions should be changed with the first terms of their Taylor series with respect to the conditions (17). The form of the equations obtained is:

$$\mathfrak{f}'' + \frac{\mathfrak{f}'}{r} - \left(2\mu^2 - \omega_1^2\right)\mathfrak{f} = 0; \quad \mathfrak{g}'' + \frac{\mathfrak{g}'}{r} - \left(2\mu^2 - (\omega_1 - \omega_2)^2\right)\mathfrak{g} = 0 \quad (18)$$

Or in operator formalism:

$$\left[\Delta_r - \left(2\mu^2 - \omega_1^2\right)\right]\mathfrak{f} = 0; \quad \left[\Delta_r - \left(2\mu^2 - (\omega_1 - \omega_2)^2\right)\right]\mathfrak{g} = 0 \quad (19)$$

Where Δ_r is radial component of Laplace operator in polar coordinates. One can notice that these are Klein-Gordon-Fock equations for radially symmetrical massive scalar fields. For the solutions to be localized [3], the mass parameters must both be positive, therefore the system of inequalities for angular frequences takes place

$$\begin{cases} \omega_1 \le \sqrt{2}\mu \\ \omega_2 \le \sqrt{2}\mu + \omega_1 \end{cases} \quad (20)$$

This system imposes a trapezoid area of stable localized solitonic configurations.

2 Methods and results

The main aim of the research was to construct a method of getting profile functions F and G with the set of parameters $\omega_1, \omega_2, n_1, n_2$ and then calculate the integrals of dynamical characteristics corresponding to each of the pairs of F, G. For the sake of finitness of the numerical integral sum the coordinate r was compactified $r \to r/\,(1 + r) \in [0,\ 1)$ to the unit segment. The conjugate gradient method was used for the action with Lagrangian (3) minimization. Derivatives were approximated by finite-difference scheme with number of grid nodes $n = 90$ on a $[0,\ 1)$ unit segment. Mesh 50×100 was chosen on the angular frequency domain.

2.1 Profile functions

Two samples of profile functions obtained by finite-difference scheme Lagrangian minimization (see Figure 1.1)

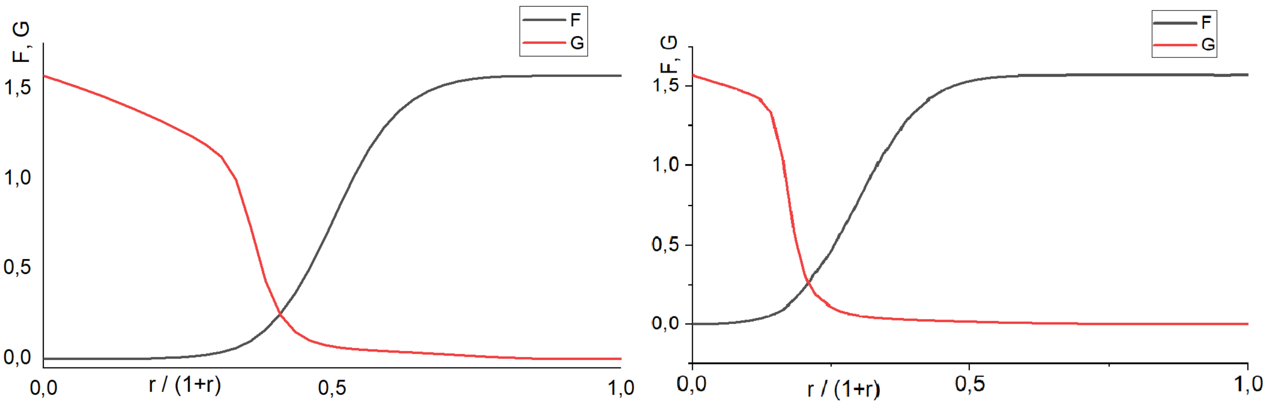

Figure 1.1: Profile functions
Left plot: $\omega_1 = 1.0,\ \omega_2 = 2.0,\ n_1 = 3,\ n_2 = 4$; Right plot: $\omega_1 = 5.0,\ \omega_2 = 6.0,\ n_1 = 1,\ n_2 = 2$

2.2 Solutions stability

It must be mentioned, that correctness of constraints (20) was affirmed by numerical study. Let's consider a full domain frequency space (see Figure 2.1, left). The regions of numerical convergence of action belong to the colored gradient part of the plot. The regions of action divergence, in turn, are depicted as grey regions. As one can notice, both of the regions are continuos and have distinct border, which has a form of a trapezoid (20). Thus, hereinafter all the plots of dynamical characteristics will be reduced to stable region (see Figure 2.1, right)

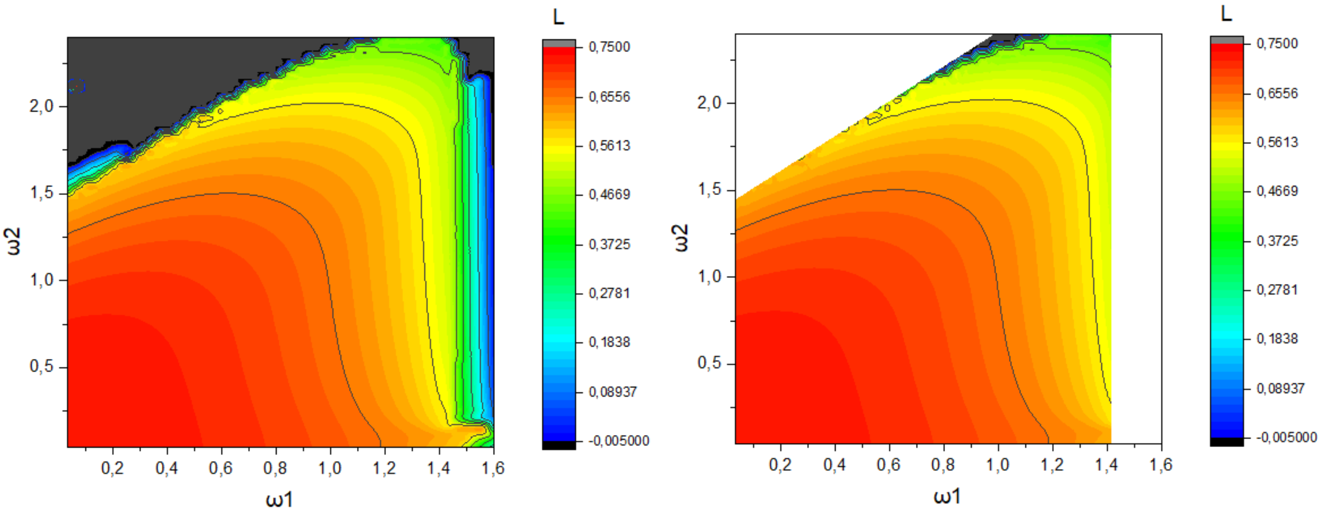

Figure 2.1: Domain of action convergence

2.3 Dynamical characteristics plots

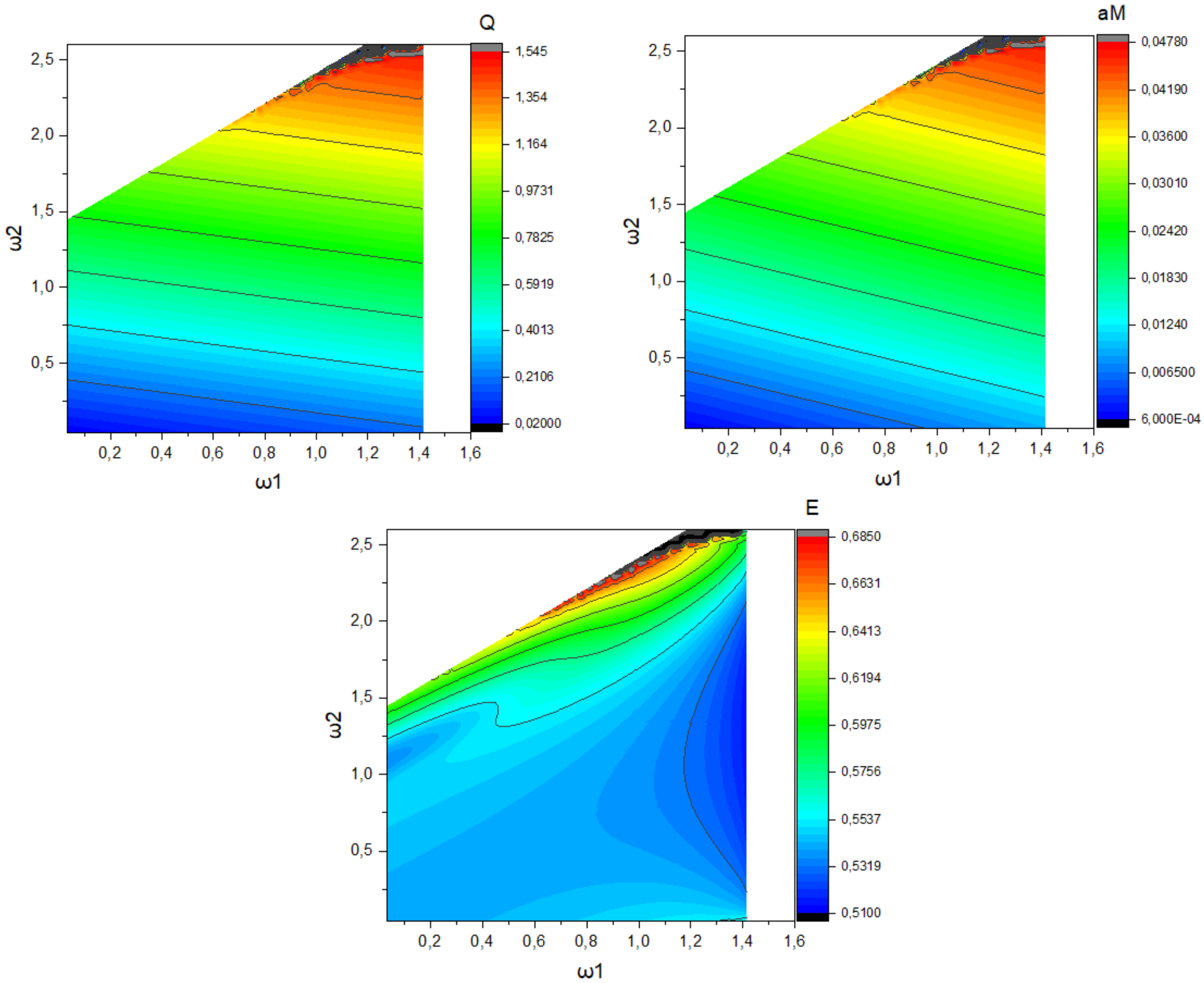

Figure 3.1: $n_1 = 2,\ n_2 = 2$
Total Noether charge (top left plot), total angular momentum (top right plot) and energy (bottom plot)

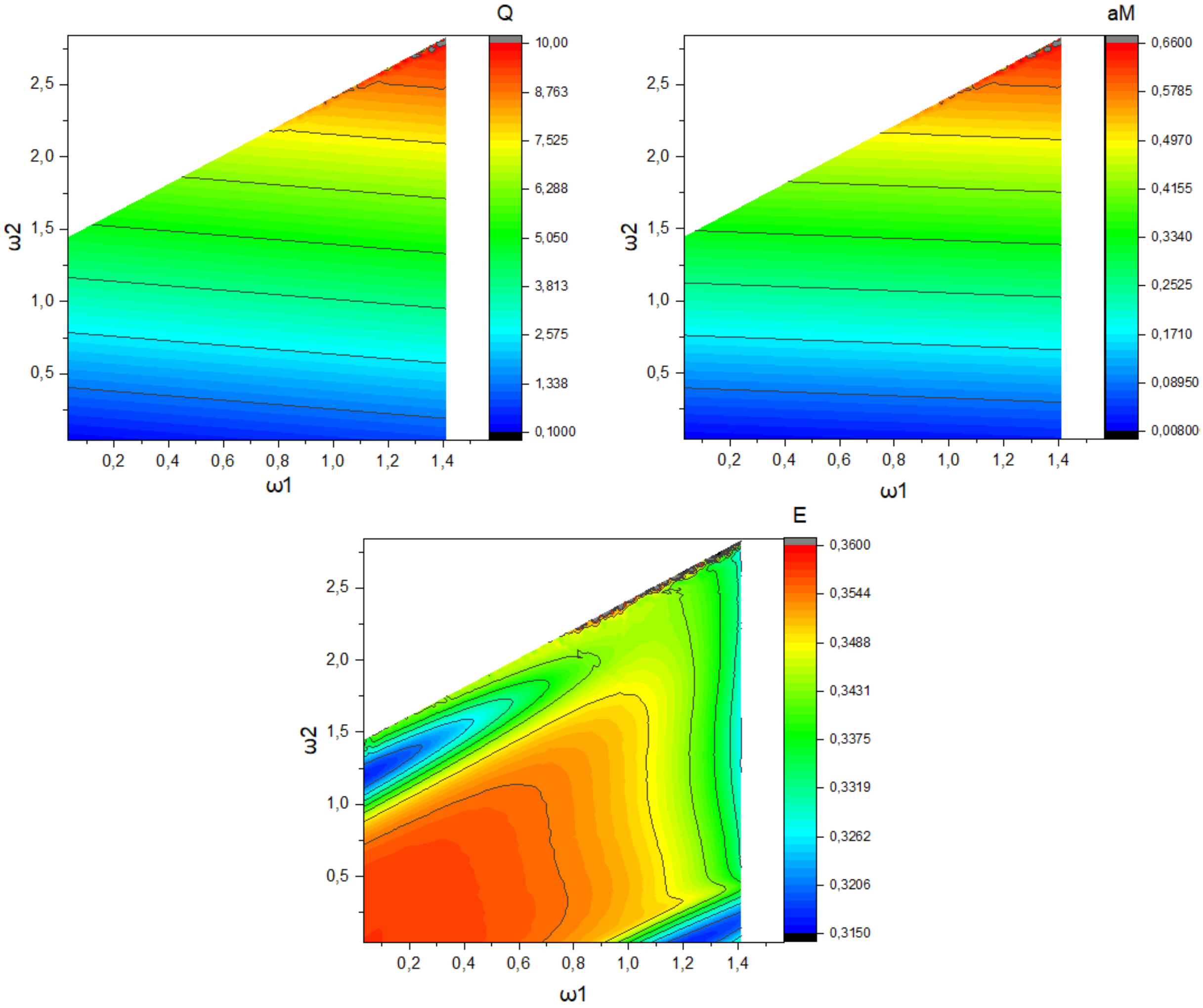

Figure 3.2:$n_1 = 1,\ n_2 = 4$
Total Noether charge (top left plot), total angular momentum (top right plot) and energy (bottom plot)

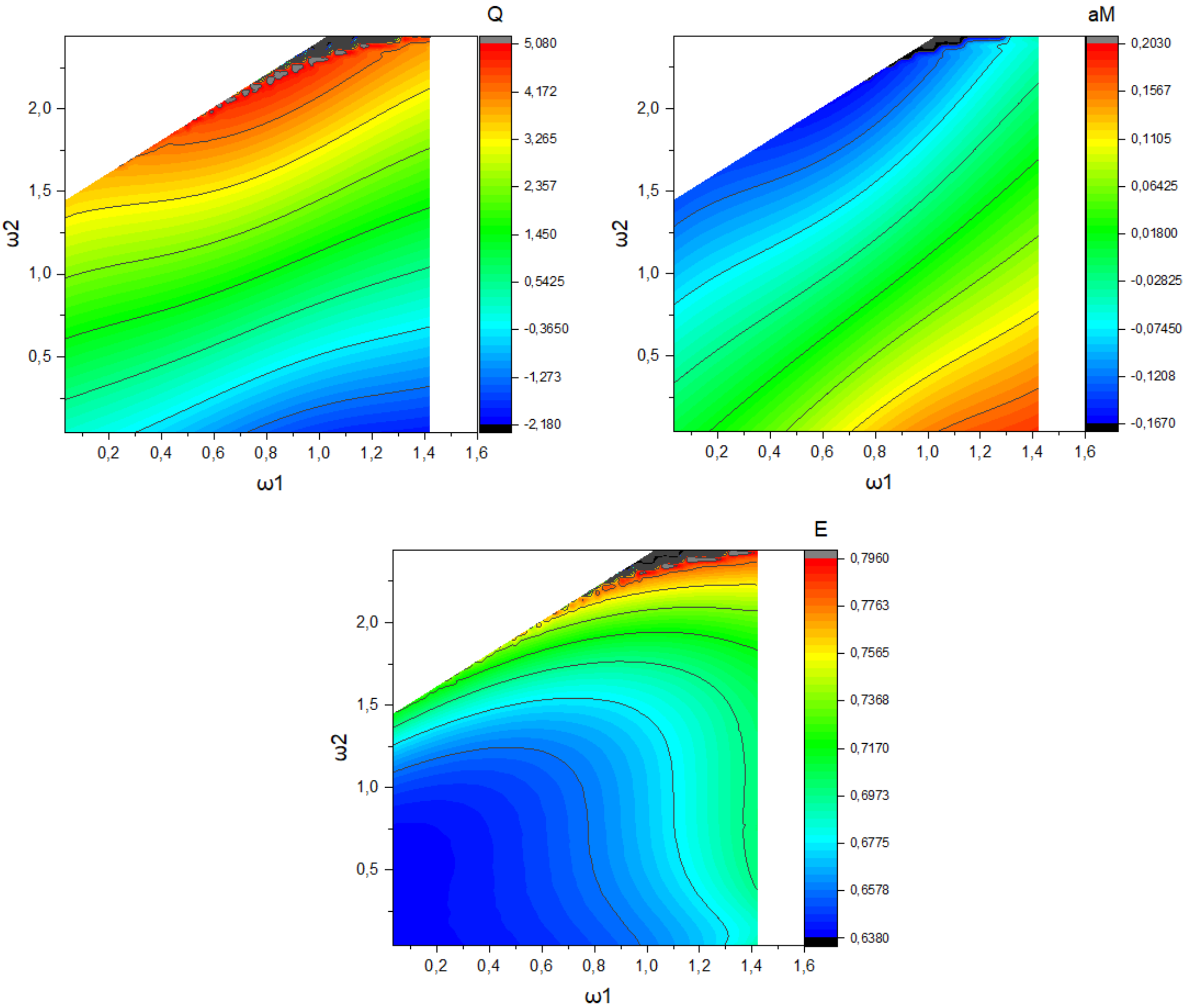

Figure 3.3: $n_1 = 3,\ n_2 = 1$
Total Noether charge (top left plot), total angular momentum (top right plot) and energy (bottom plot)

3 Conclusion

In this research the isorotation properties (i.e. internal rotations) of $\mathbb{C}P^2$ monosolitonic solutions in (2+1) dimensions with classical Lagrangian were investigated. At first the theoretical outline of the model was carried out, the suitable ansatz was chosen. Then the explicit expressions for the dynamical characteristics were found. The obtained plots of action and energy clearly show the stability domain, this fact approves the theoretical results. In our next research the problems of angular momenta and charges relations will be taken up for more systematic investigation.

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